k patches

n species 

k patches

n species 

r_i^ℓ intrinsic rate of growth of i in ℓ

a_{ij}^ℓ per-capita effect of j on i in ℓ

x_i^ℓ density of i in ℓ

$E_i^\ell(t)$ Brownian motion for i in ℓ

$$dx_i^\ell = x_i^\ell\left\{\Big(\overbrace{\underbrace{\sum_{j=1}^n a_{ij}^\ell x_j^\ell + r_i^\ell}_{=:\, f_i^\ell(x^\ell)}}^{\text{Lotka-Volterra}}\Big)dt + dE_i^\ell\right\}$$

k patches

n species 

r_i^ℓ intrinsic rate of growth of i in ℓ

a_{ij}^ℓ per-capita effect of j on i in ℓ

p_i^ℓ proportion of spp. i in patch ℓ

$x_i^\ell = p_i^\ell x_i$ where x_i regional density

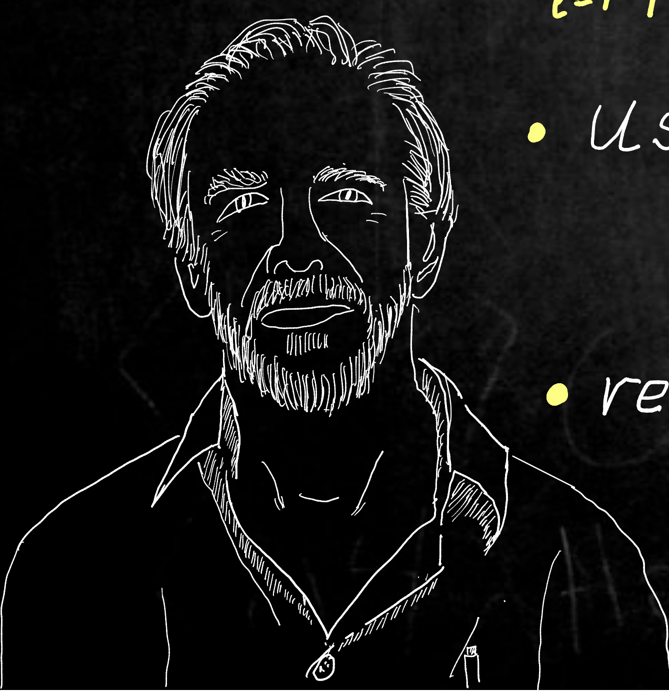
$(E_i^1(t), \ldots, E_i^k(t))$ Brownian motion for i with covariance matrix $(\sigma_i^{\ell m})_{\ell, m}$

$$\boxed{dx_i = x_i \sum_{\ell=1}^{k} p_i^\ell \left\{ f_i^\ell (p^\ell \circ x^\ell)\, dt + dE_i^\ell \right\}}$$

$$dx_i = x_i \sum_{\ell=1}^{k} p_i^{\ell} \Big\{ \overbrace{\Big(\sum_{j=1}^{n} a_{ij}^{\ell} p_j^{\ell} x_j + r_i^{\ell}\Big)}^{\text{Lotka-Volterra}} dt + dE_i^{\ell} \Big\}$$

an overdressed Lotka Volterra SDE

$$dx_i = x_i \Big\{ \Big(\sum_{j=1}^{n} \underbrace{\tilde{a}_{ij}}_{= \sum_{\ell=1}^{k} p_i^{\ell} p_j^{\ell} a_{ij}^{\ell}} x_j + \tilde{r}_i \Big) dt + \underbrace{dW_i}_{= \sum_{\ell=1}^{k} p_i^{\ell} dE_i^{\ell}} \Big\}$$



- Use the Hofbauer criterion to determine whether resident spp. coexist.
- reduces to solving systems of linear equations & linear programming

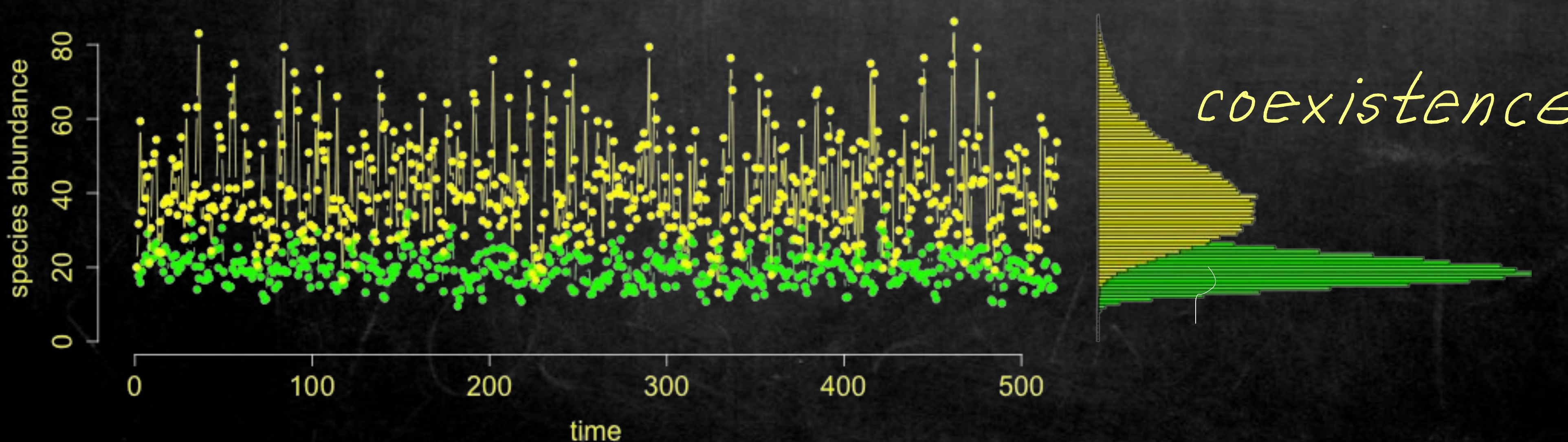
[S. et al. 2011, Hening & Nguyen 2018]

Coexistence = a probabity measure $\mu(dx)$ s.t.

- $\mu((0,\infty)^n) = 1$
- for $\varphi \in \mathcal{L}^1(\mu)$ and $x_i(0) > 0$ for all i

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(x(s))\,ds = \int \varphi(x)\,\mu(dx)$$

with prob. one



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with prob. one

Coexistence criteria imply

- $\bar{x}_i := \int x_i \mu(dx) < +\infty$
- $0 = \sum_j \tilde{a}_{ij}\,\bar{x}_j - r_i - \frac{1}{2}\sum_{l,m} p_i^l p_i^m \sigma_i^{lm}$

i.e. $\bar{x}_i$ are a soln of a system of linear eqns

$$dx_i = x_i \sum_{\ell=1}^{k} p_i^{\ell} \left\{ f_i^{\ell}(p^{\ell} \circ x^{\ell})\,dt + dE_i^{\ell} \right\}$$

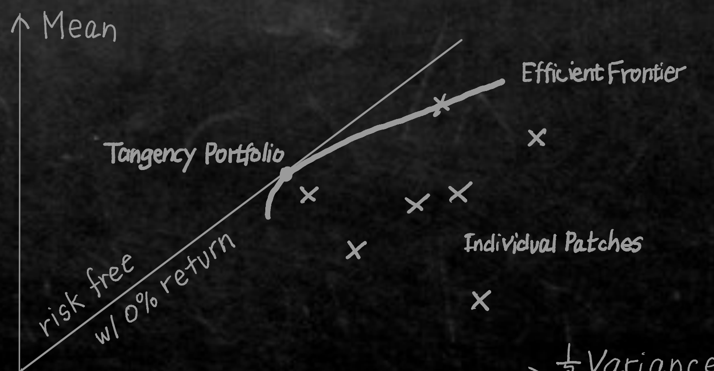
I. Models & Coexistence

II. Main Results: Coevolutionary Stability


III. Applications: Competing Species


Predator – Prey

IV. Modern Portfolio Theory & Finale



Evolutionary Stable Strategy

resident community playing p, mean densities $\bar{x}$

mutant of species $\tilde{i}$ playing q, density y

$$dx_i = x_i \sum_{\ell=1}^{k} p_i^\ell \left\{ \left(f_i^\ell(p^\ell \circ x^\ell) + a_{i\tilde{i}}\, q^\ell y \right) dt + dE_i^\ell \right\}$$

$$dy = y \sum_{\ell=1}^{k} q^\ell \left\{ \left(f_{\tilde{i}}^\ell(p^\ell \circ x^\ell) + a_{\tilde{i}\tilde{i}}\, q^\ell y \right) dt + dE_{\tilde{i}}^\ell \right\}$$

invasion rate

$$\mathcal{I}_{\tilde{i}}(p,q) = \sum_{\ell=1}^{k} q^\ell\, f_{\tilde{i}}^\ell(p^\ell \circ \bar{x}^\ell) - \frac{1}{2} \sum_{\ell,m=1}^{k} q^\ell q^m \sigma_{\tilde{i}}^{\ell m}$$

Theorem $\mathcal{I}_{\tilde{i}}(p,q) < 0 \Longrightarrow$

$$\mathbb{P}\left[\lim_{t\to\infty} y(t) = 0\right] \uparrow 1 \text{ as } y(0) \downarrow 0$$

p is a coevolutionarily stable strategy

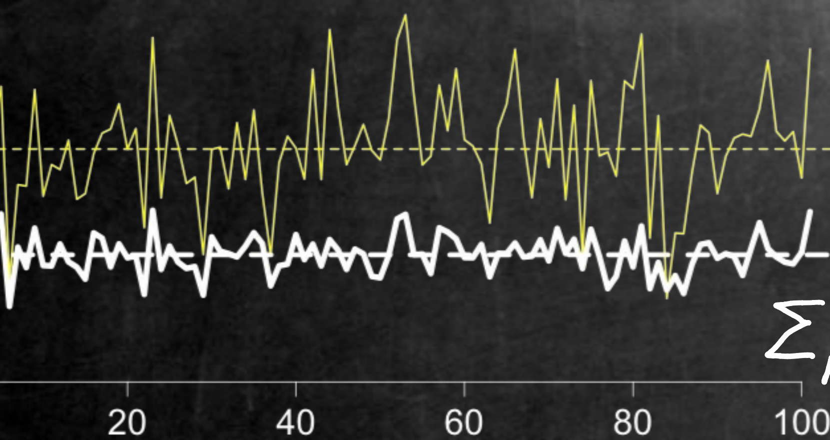
(coESS)

if $\mathcal{L}_{\tilde{i}}(p,q) < 0$ for all i and $[q^1 \dots q^k] \neq p_{\tilde{i}}$

$$p_{\tilde{i}} = [p^1_{\tilde{i}} \dots p^k_{\tilde{i}}]$$

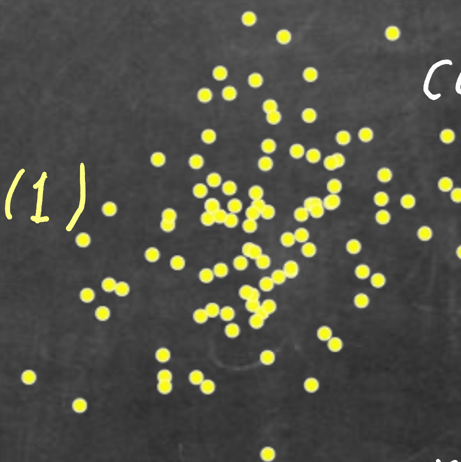
Characterization of coESS

mean regional growth rate $M_i = \sum_{\ell=1}^{k} p_i^\ell f_i^\ell(p_0^\ell \bar{x}^\ell)$



$E_i^\ell(t)$

$\sum_m p_i^m E_i^m(t)$



covariance cov_i^ℓ

$E_i^\ell(1)$

$\sum_m p_i^m E_i^m(1)$

regional environmental variance

$$V_i = \sum_{\ell=1}^{k} p_i^\ell \, cov_i^\ell$$

Proposition at coESS

$$\boxed{\begin{array}{l} f_i^\ell - cov_i^\ell = M_i - V_i \text{ in occupied } \ell \\ f_i^\ell - cov_i^\ell < M_i - V_i \text{ in unoccupied} \end{array}}$$

$$\boxed{\begin{array}{l} f_i^{\ell} - cov_i^{\ell} = M_i - V_i \text{ in occupied } \ell \\ f_i^{\ell} - cov_i^{\ell} < M_i - V_i \text{ in unoccupied} \end{array}}$$

$$M_i = \sum_{\ell=1}^{k} p_i^{\ell} f_i^{\ell}(p_0^{\ell} \bar{x}^{\ell}) \quad V_i = \sum_{\ell=1}^{k} p_i^{\ell} cov_i^{\ell}$$

Implications

- local stochastic growth rates $f_i^{\ell} - \frac{1}{2}\sigma_i^{\ell\ell}$ typically not equal among occupied ℓ
- either only one patch occupied

  or

  all patches are sinks i.e. $f_i^{\ell} - \frac{1}{2}\sigma_i^{\ell\ell} < 0$