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EXPANSION RATES AND LYAPUNOV EXPONENTS

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Abstract. The logarithmic expansion rate of a positively invariant set for a C^1 endomorphism is shown to equal the infimum of the Lyapunov exponents for ergodic measures with support in the invariant set. Using this result, aperiodic flows of the two torus are shown to have an expansion rate of zero and the effects of conjugacies on expansion rates are investigated.

1. Introduction

Let f be a continuous endomorphism of a Riemannian manifold M for which the compact subset, $K \subset M$, is positively invariant (*i.e.*, $f(K) \subset K$). Define the *expansion constant of f at K*, $EC(f,K)$, to be the largest value $\rho \geq 0$ that satisfies: given $0 < \rho^* < \rho$ there exists a $\delta > 0$ such that $d(fx, fy) \geq \rho^* d(x,y)$ for all $x \in K$ and $y \in M$ where $d(x,y) \leq \delta$. When f is C^1,

$$EC(f,K) = \min_{x \in K} m(Df_x)$$

where $m(Df_x) = \inf_{|v|=1} |Df_x v|$. Although f might not be expansive immediately, eventually it can be. Taking this into account, Hirsch [1] defined the *logarithmic expansion rate of f at K* to be the number given by

$$\mathcal{E}(f,K) = \sup_{n \geq 0} \frac{1}{n} \log EC(f^n, K).$$

The purpose of this manuscript is to answer three questions posed by Hirsch [1, 2]:

(1) How does the expansion rate of a C^1 endomorphism relate to the Lyapunov exponents of the derivative cocycle?
(2) Do all aperiodic flows of the two-torus have logarithmic expansion rate zero?
(3) How are expansion rates of topologically conjugate endomorphisms related?

In section 1, the statement of main theorem is presented answering the first question. In section 2, several corollaries of main theorem are proven. In particular for $C^{1+\alpha}$ ($\alpha > 0$) endomorphisms we show that exponentially attracted or repelled points characterize the expansion rate. Using this we are able to answer question 2. In section 3, question 3 is addressed. In section 4, a Lipschitz flow of the unit interval is constructed with following property: the expansion rate of the flow is strictly negative but there exist no exponentially attracted points. In section 5, the proof of the main theorem is given.

2. Main Result

Given μ be a Borel probability measure for which f is ergodic, define the *logarithmic expansion rate of f with respect to μ*, $\mathcal{E}(f,\mu)$, to be the smallest Lyapunov exponent for f and μ. The multiplicative ergodic theorem [7] asserts there exists a Borel set, $\mathcal{R}$, such that $\mu(\mathcal{R})=1$ and for all $x\in\mathcal{R}$

$$\lim_{n\to\infty}\frac{1}{n}\log m(Df^n_x)$$

exists and equals $\mathcal{E}(f,\mu)$.

Theorem 1. *Let f be a C^1 endomorphism of a Riemannian manifold M for which the compact set K is positively invariant. If f is immersive at K (i.e., $\det(Df_x)\neq 0$ for all $x\in K$), then*

$$\mathcal{E}(f,K)=\inf_\mu \mathcal{E}(f,\mu)$$

where the infimum is taken oven all ergodic measures with support K. Equivalently,

$$\mathcal{E}(f,K)=\inf_{x\in K}\liminf_{n\to\infty}\frac{1}{n}\log m(Df^n_x)$$

To see the necessity of the additional hypothesis, f is immersive at K, consider a C^1 homeomorphism of S^1, f, such that f is conjugate to the north pole-south pole flow and has a unique critical point distinct from the north and south pole. For this map, $\mathcal{E}(f,S^1)=-\infty$. However, since the only invariant ergodic measures for f are the Dirac measures at the north or south pole, $\inf_\mu\mathcal{E}(f,\mu)=\log|f'(\text{southpole})|>-\infty$.

Theorem 1 shows for which subset of K, it is sufficient to check expansiveness. Let $BC(f,K)$ denote the Birkhoff center of f restricted to K (*i.e.*, the closure of the recurrent points in K). Hirsch, Pugh and Shub [4] have shown that

$$\mathcal{E}(f,K)=\mathcal{E}(f,BC(f,K)).$$

Theorem 1 asserts

$$\mathcal{E}(f,K)=\mathcal{E}(f,M(f)\cap K).$$

where $M(f)$ is the smallest closed set such that $\mu(M(f))=1$ for every f-invariant Borel probability measure μ on M. The Poincaré recurrence theorem implies $\mathcal{M}(f)\cap K\subset BC(f,K)$. However, in general, the opposite inclusion is not true. For instance, Nemytskii and Stepanov [6] constructed a diffeomorphism, f, of the two-torus, T^2, such that $BC(f,T^2)=T^2$ yet $M(f)$ is a single point.

3. Corollaries of Theorem 1

The first corollary of Theorem 1 is immediate from the metric invariance of Lyapunov exponents.

Corollary 1. *Let f and K be as in Theorem 1. Then $\mathcal{E}(f,K)$ is independent of the Riemannian metric on M.*

Using results in smooth ergodic theory, more corollaries of Theorem 1 are derived. Recall, given an endomorphism, $f:M\to M$, we can define the inverse limit, $\tilde{f}^{-1}:\tilde{M}\to\tilde{M}$, as follows: Let $\tilde{M}$ be the sequence space $\{(x_n)_{n\geq 0}\in M: f(x_{n+1})=x_n\}$ endowed with the metric, $\tilde{d}(\tilde{x},\tilde{y})=\sup_n d(x_n,y_n)$, where $\tilde{x}=(x_n),\tilde{y}=(y_n)\in\tilde{M}$ and d is the metric on M. $\tilde{M}$ is the space of "orbital histories" for f. Define $\tilde{f}^{-1}:\tilde{M}\to\tilde{M}$ by $(x_n)_{n\geq 0}\mapsto(x_n)_{n\geq 1}$. If f is continuous and M is compact then $\tilde{M}$ is a compact metric space and $\tilde{f}^{-1}$ is a homeomorphism with inverse, $\tilde{f}(x_n)=(fx_n)$. Let $\pi:\tilde{M}\to M$ be the projective map given by $(x_n)\mapsto x_0$. Given $\tilde{x}\in\tilde{M}$ and $\delta>0$, define $B(\tilde{x},\delta)=\{\tilde{y}\in\tilde{M}$ such that $\tilde{d}(\tilde{x},\tilde{y})\leq\delta\}$.

Corollary 2. *Let f and K be as in Theorem 1. Let $\lambda > 0$ be given.*

(1) *Assume f is $C^{1+\alpha}$ for some $\alpha > 0$. Then $\mathcal{E}(f,K) < -\lambda$ if and only if there exists $x \in K$ and $y \in M \backslash x$ such that*
$$\limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) < -\lambda. \tag{1}$$

(2) *Assume f is C^1 and is bi-invariant (i.e., $f(K) = f^{-1}(K) = K$). Then $\mathcal{E}(f,K) > \lambda$ if and only if there exists a $\delta_{\tilde{x}} > 0$ for each $\tilde{x} \in \pi^{-1}(K)$ such that $\pi B(\tilde{x}, \delta_{\tilde{x}})$ contains a neighborhood of $\pi(\tilde{x})$ and*
$$\limsup_{n\to\infty} \frac{1}{n} \log \tilde{d}(\tilde{f}^n B(\tilde{x}, \delta_{\tilde{x}}), \tilde{f}^n \tilde{x}) < -\lambda.$$

Proof. Assume $\mathcal{E}(f,K) < -\lambda$. The stable manifold theorem for endomorphisms (see for example [8]) implies there exist exponentially attracted points satisfying Equation 1. The other direction of 1 follows immediately from the topological definition of the logarithmic expansion rate.

Assume $\mathcal{E}(f,K) > \lambda$. Then the implication to the right in 2 can be derived from the topological definition of logarithmic expansion rate. On the other hand, if the right hand side of 2 holds, then all the Lyapunov exponents of f are greater than λ, and therefore by Theorem 1 $\mathcal{E}(f,K) > \lambda$. □

Remark. (1) The point of this corollary is to quantify the expansion rate by non-uniform pointwise properties. (2) Notice that each assertion has an implication that only requires continuity. However, continuity is not sufficient for all implications. For instance, in section 4 we construct a Lipschitz flow of $[0,1]$ with a negative logarithmic expansion rate, yet no exponentially attracted points. Consequently, we pose the following question:

Question. *Let f be a C^1 endomorphism that is immersive at the positively invariant set K. If $\mathcal{E}(f,K) < 0$, does there exist $x \in K$ and $y \in M \backslash x$ such that Equation 1 holds for some $\lambda > 0$?*

The second corollary of Theorem 1 calculates the expansion rate for a negatively Lyapunov stable set. Assume f is a homeomorphism of M. A point $x \in M$ is called *negatively Lyapunov stable* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f^n x, f^n y) < \epsilon$ for all $n \leq 0$. For instance, if f is minimal and almost periodic, then each $x \in M$ is negatively Lyapunov stable [6].

Corollary 3. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold M for which the compact set K is positively invariant. If every $x \in K$ is negatively Lyapunov stable, then $\mathcal{E}(f,K) \geq 0$.*

Proof. As x is negatively Lyapunov stable for all $x \in K$ and K is compact, a standard covering argument shows that K is uniformly negatively Lyapunov stable: given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(p,q) < \delta$, $p \in K$, $q \in M$, implies $d(f^n p, f^n q) < \epsilon$ for all $n \leq 0$. Suppose $\mathcal{E}(f,K) < 0$. Pick $\lambda < 0$ such that $\mathcal{E}(f,K) < -\lambda$. By Corollary 2 there exists $x \in K$ and $y \in M \backslash x$, such that Equation 1 holds. Choose $\epsilon < d(x,y)/2$. Given any $\delta > 0$, Equation 1 implies there exists $N > 0$ such that $d(f^N x, f^N y) < \delta$. Setting $p = f^N x$ and $q = f^N y$ leads to a contradiction. □

Using Corollary 2, negative logarithmic expansion rates of C^1 endomorphisms of one-dimensional manifolds are easy to characterize.

Corollary 4. *Let f be a C^1 endomorphism of S^1 or $\mathbf{R}$ such that the compact set K is positively invariant. Let $\lambda > 0$ be given. If f is immersive at K and $\mathcal{E}(f,K) < -\lambda$, then there exists a periodic orbit of f in K with its Lyapunov exponent $< -\lambda$.*

Proof. Choose an ergodic measure, μ, such that $\mathcal{E}(f,\mu) < -\lambda$. For one dimensional maps this implies that μ is concentrated on a periodic point whose Lyapunov exponent must be $< -\lambda$ (for details see [9]). $\square$

In particular, Corollary 4 implies if f has no periodic points in K, then $\mathcal{E}(f,K) \geq 0$ (*e.g.*, f is a C^1 diffeomorphism of $S^1 = K$ with irrational rotation number). This provides an affirmative answer to the second question posed in the introduction.

4. Effects of conjugation

Proposition 1. *Let f and K be as in Theorem 1. Let $g = h \circ f \circ h^{-1}$ where h is a C^α ($\alpha \in [0,1]$) homeomorphism of M.*

1. *Assume f is $C^{1+\beta}$ for some $\beta > 0$. If $\mathcal{E}(f,K) < 0$ then $\mathcal{E}(g,hK) \leq \alpha\mathcal{E}(f,K)$.*
2. *Assume f is C^1 and K is bi-invariant. If $\mathcal{E}(f,K) > 0$ then $\mathcal{E}(g,hK) \geq \alpha\mathcal{E}(f,K)$.*

Proof. To prove 1, let $\lambda > 0$ be given. Suppose there exist $x \in K$ and $y \in M \backslash x$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) \leq -\lambda.$$

Then

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log d(g^n hx, g^n hy) &= \limsup_{n\to\infty} \frac{1}{n} \log d(h\circ f^n x, h \circ f^n y) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y)^\alpha = -\lambda\alpha\end{aligned}$$

This in conjunction with Corollary 2 implies $\mathcal{E}(g,hK) \leq \alpha\mathcal{E}(f,K)$.

To prove 2, let $\tilde{M}_f = \{(x_n)_{n\geq 0} : fx_{n+1} = x_n\}$ and $\tilde{M}_g = \{(x_n)_{n\geq 0} : gx_{n+1} = x_n\}$. Define $\pi_f : \tilde{M}_f \to M$ and $\pi_g : \tilde{M}_g \to M$ by $(x_n) \mapsto x_0$. On $\tilde{M}_f$ and $\tilde{M}_g$, we place the metric $\tilde{d}((x_n),(y_n)) = \sup_n d(x_n,y_n)$. Define $\tilde{h} : \tilde{M}_f \to \tilde{M}_g$ by $(x_n) \mapsto (h(x_n))$. $\tilde{h}$ is well defined as $g(h(x_{n+1})) = h(f(x_{n+1})) = h(x_n)$ for all $(x_n) \in \tilde{M}_f$. Furthermore, $\tilde{h}$ is a C^α homeomorphism. Assume $\mathcal{E}(f,K) > \lambda > 0$. Let $\tilde{x} \in \pi_f^{-1}(K)$ be given. By Corollary 2 there exists $\delta > 0$ such that $\pi_f(B(\tilde{x},\delta))$ contains a neighborhood of $\pi_f(\tilde{x})$ in M and

$$\limsup_{n\to\infty} \frac{1}{n} \log \tilde{d}(\tilde{f}^{-n}(B(\tilde{x},\delta)), \tilde{f}^{-n}\tilde{x}) < -\lambda.$$

As $\tilde{h}$ and h are homeomorphisms, there exist $\delta_0 > 0$ such that $B(\tilde{h}\tilde{x},\delta_0) \subset \tilde{h}(B(\tilde{x},\delta))$ and $\pi_g B(\tilde{h}\tilde{x},\delta_0)$ contains a neighborhood of $\pi_g \tilde{h}\tilde{x}$ in M. As $\tilde{h}$ is C^α

and $\tilde{g}^{-1} = \tilde{h} \circ \tilde{f}^{-1} \circ \tilde{h}^{-1}$, we get

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log \tilde{d}(\tilde{g}^{-n}(B(\tilde{h}\tilde{x}, \delta_0)), \tilde{g}^{-n}\tilde{h}\tilde{x}) &\leq \limsup_{n\to\infty} \frac{1}{n} \alpha \log \tilde{d}(\tilde{f}^{-n}(B(\tilde{x}, \delta), \tilde{f}^{-n}\tilde{x}) \\ &< -\alpha\lambda. \end{aligned}$$

Corollary 2 implies that $\mathcal{E}(g, hK) > \alpha\mathcal{E}(f, K)$. □

5. A counterexample

In this section, we provide the example alluded to in the remark following Corollary 2.

Proposition 2. *Let $\lambda > 0$ be given. Then there exists a Lipschitz map $f : [0, 1] \to [0, 1]$ such that*

(1) *$\mathcal{E}(\phi_1, [0, 1]) \leq -\lambda$ where ϕ_t is the flow generated by $\dot{x} = f(x)$.*
(2) *For all $x, y \in [0, 1]$, $x \neq y$, the following equation holds*

$$\lim_{t\to\infty} \frac{1}{t} \log d(\phi_t x, \phi_t y) = 0.$$

Proof. Let $a = e^{-\lambda}$, $b_0 = 1$ and $b_{n+1} = a^{n+2} b_n$. Choose a smooth function $g : [a, 1] \to \mathbf{R}$ such that $g(1) = g'(1) = 0$, $g(a) = -\lambda a$, $g'(a) = -\lambda$ and $g(x) < 0$ for all $x \in [a, 1)$. Let $h : [a, 1] \to \mathbf{R}$ be a smooth function that satisfies $h(a) = h'(a) = 0$, $h(1) = h'(1) = -\lambda$ and $h(x) < 0$ for all $x \in [a, 1)$.

Define f as follows,

$$f(x) = \begin{cases} b_n g(x/b_n) & x \in [ab_n, b_n] \\ -\lambda x & x \in [a^{n+1}b_n, ab_n] \\ b_n a^{n+1} h(x/(a^{n+1}b_n)) & x \in [b_{n+1}, b_n a^{n+1}] \\ 0 & x = 0 \end{cases}$$

Since we are just rescaling three smooth maps, f is Lipschitz. However, f is clearly not C^1 at 0. Let ϕ_t be the flow generated by $x'(t) = f(x(t))$. Since $f(x) = -\lambda x$ on the intervals $[a^{n+1}b_n, ab_n]$, it follows that

$$\phi_t(ab_n) = ab_n e^{-\lambda t} = a^{t+1} b_n$$

for all $t \in [0, n]$. Therefore

$$\mathcal{E}(\phi_1, [0, 1]) \leq -\lambda.$$

By construction the b_n are saddle nodes. Therefore for each $x \in (0, 1]$, $\phi_t(x) \to b_n$ at a subexponential rate as $t \to \infty$ for some n. This proves the second assertion of the proposition. □

6. Proof of Theorem 1

Let

$$E = \inf_{\mu} \mathcal{E}(f, \mu)$$

where the infimum is taken over all ergodic measures with support in K.

First, we show $E \leq \mathcal{E}(f, K)$. By definition of $\mathcal{E}(f, K)$, for each n there is a point, $x_n \in K$, and a unit vector, $v_n \in T_{x_n} M$, such that

$$\frac{1}{n} \log |Df^n_{x_n} v_n| \leq \mathcal{E}(f, K). \tag{2}$$

Let $T^1_K M$ denote the unit tangent bundle restricted to K. Define a continuous endomorphism of $T^1_K M$, by

$$g(x,v) = \left(fx, \frac{Df_x v}{|Df_x v|} \right).$$

which is well defined as $\det Df_x \neq 0$ for all $x \in K$. Define a sequence of Borel probability measures on $T^1_K M$ by

$$\mu_n = \frac{1}{n} \sum_{0}^{n-1} \delta_{g^i(x_n,v_n)}$$

where $\delta_{(x,v)}$ is the Dirac measure at the point (x,v). By compactness of $T^1_K M$, there is a subsequence, μ_{n_k}, that weakly converges to a limit, μ. Given any continuous function, $h : T^1_K M \to \mathbf{R}$, weak convergence and boundedness of h implies

$$\begin{aligned} \int h(gx) d\mu(x) &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{0}^{n_k-1} h(g^{i+1} x_{n_k}) \\ &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{0}^{n_k-1} h(g^i x_{n_k}) - \lim_{k\to\infty} \frac{1}{n_k} \left(h(x_{n_k}) - h(g^{n_k} x_{n_k}) \right) \\ &= \int h(x) d\mu(x). \end{aligned}$$

Hence μ is g-invariant.

Define $h : T^1_K M \to \mathbf{R}$ by $(x,v) \mapsto \log |\pi Df(x) v|$. Notice that

$$\begin{aligned} \sum_{0}^{n-1} h(g^i(x,v)) &= \log |Df_x v| \frac{|Df^2_x v|}{|Df_x v|} \frac{|Df^3_x v|}{|Df^2_x v|} \cdots \frac{|Df^n_x v|}{|Df^{n-1}_x v|} \\ &= \log |Df^n_x v|. \end{aligned} \tag{3}$$

Weak convergence, and Equations 3 and 2 imply

$$\begin{aligned} \int h(x) d\mu(x) &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{0}^{n_k-1} h(g^i(x_n, v_n)) \\ &= \lim_{k\to\infty} \frac{1}{n_k} \log m(Df_{x_{n_k}} v_{n_k}) d\mu(x) \\ &\leq \mathcal{E}(f,K) \end{aligned} \tag{4}$$

By the ergodic decomposition theorem, we may assume that μ is an ergodic measure that satisfies Equation 4. Birkhoff's ergodic theorem implies there exists an g-invariant Borel set, $\mathcal{R} \subset T^1_K M$, such that $\mu(\mathcal{R}) = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n_x v| \leq \mathcal{E}(f,K)$$

for all $(x,v) \in \mathcal{R}$.

Define $\pi : T^1_K M \to K$ by $\pi(x,v) = x$ and let $\tilde{\mu} = \mu \circ \pi^{-1}$. $\tilde{\mu}$ is f-invariant, has support in K, and makes f ergodic. $\pi\mathcal{R}$ has full $\tilde{\mu}$ measure. And for each $x \in \pi\mathcal{R}$ there exists $v \in T_x M$, $|v| = 1$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log m\left(Df^n_x\right) \leq \lim_{n\to\infty} \frac{1}{n} \log |Df^n_x v| \leq \mathcal{E}(f,K).$$

Therefore,

$$E \leq \mathcal{E}(f, \tilde{\mu}) \leq \mathcal{E}(f, K).$$

To prove $E \geq \mathcal{E}(f, K)$ is much easier. Pick $\epsilon > 0$. There exists a measure μ with support in K for which f is ergodic such that $\mathcal{E}(f, \mu) - \epsilon \leq E$. By the subadditive ergodic theorem there exists an x in K such that

$$\lim_{n \to \infty} \frac{1}{n} \log m\left(Df_x^n\right) = \mathcal{E}(f, \mu).$$

Pick N sufficiently large such that

$$\frac{1}{n} \log m(Df_x^n) \leq \mathcal{E}(f, \mu) + \epsilon \leq E + 2\epsilon$$

for all $n \geq N$. Given $k \in \mathbf{N}$,

$$\begin{aligned} E + 2\epsilon &\geq \frac{1}{Nk} \log m(Df_x^{kN}) \\ &\geq \frac{1}{Nk} \left(\log m(Df_x^k) + \dots + \log m(Df^k_{f^{kN-k}x}) \right). \end{aligned}$$

Therefore there exists a $0 \leq n \leq N - 1$ such that

$$\frac{1}{k} \log m(Df^k_{f^{nk}x}) \leq E + 2\epsilon.$$

This implies $E + 2\epsilon \geq \mathcal{E}(f, K)$ for all $\epsilon > 0$. □

References

[1] M. W. Hirsch, *Asymptotic phase, shadowing, and reaction-diffusion systems* in "Collection: Differential equations, dynamical systems, and control science," Lecture Notes in Pure and Applied Math. 152(1994),87-99

[2] M. W. Hirsch, *personal communication* (1994).

[3] M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets*, Proc. Symp. Pure Math 14(1970),133-163

[4] M. Hirsch, C. Pugh, and M. Shub, "Invariant Manifolds," Springer Lecture Notes in Math. 583, 1977.

[5] J. F. C. Kingman, "Subadditive processes," Springer Lecture Notes in Math. 539, 1977.

[6] V. V. Nemytskii and V. V. Stepanov, "Qualitative theory of differential equations," Princeton University Press, 1960.

[7] V. I. Oseledec, *A multiplicative ergodic theorem: Lyapunov characteristic exponents for dynamical systems*, Trans. Moscow Math. Soc. 19(1960), 197-231.

[8] D. Ruelle and M. Shub, *Stable manifolds for maps* in "Collection: Global theory of dynamical systems," Springer Lecture Notes in Math. 819 (1980), 389-392.

[9] S. J. Schreiber. "Asymptotically hyperbolic dynamics," Ph.D. Dissertation, University of California, Berkeley, 1995.

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