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On Growth Rates of Subadditive Functions for Semiflows

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Let $\phi: X \times \mathbf{T}_+ \to X$ be a semiflow on a compact metric space X. A function $F: X \times \mathbf{T}_+ \to X$ is subadditive with respect to ϕ if $F(x, t+s) \leqslant F(x, t) + F(\phi(x, t), s)$. We define the maximal growth rate of F to be $\sup_{x \in X} \limsup_{t \to \infty} (1/t) F(x, t)$. This growth rate is shown to equal the maximal growth rate of the subadditive function restricted to the minimal center of attraction of the semiflow. Applications to Birkhoff sums, characteristic exponents of linear skew-product semiflows on Banach bundles, and average Lyapunov functions are developed. In particular, a relationship between the dynamical spectrum and the measurable spectrum of a linear skew-product flow established by R. A. Johnson, K. J. Palmer, and G. R. Sell (*SIAM J. Math. Anal.* **18**, 1987, 1–33) is extended to semiflows in an infinite dimensional setting. © 1998 Academic Press

1. INTRODUCTION

Consider a continuous-time or discrete-time semiflow ϕ on a compact metric space X. By this we mean a continuous map

$$\phi: X \times \mathbf{T}_+ \to X, \qquad \phi(x, t) = \phi_t x$$

that satisfies $\phi_0 x = x$ and $\phi_{s+t} x = \phi_s \phi_t x$ for $s, t \in \mathbf{T}_+$ where $\mathbf{T}_+$ equals either $\mathbf{Z}_+$ in the discrete-time case or $\mathbf{R}_+$ in the continuous-time case. An important class of continuous functions $F: X \times \mathbf{T}_+ \to \mathbf{R}$ associated with ϕ are those that satisfy a subadditivity condition:

$$F(x, t+s) \leqslant F(x, t) + F(\phi_t x, s). \tag{1}$$

These functions naturally arise in various settings including the study of Birkhoff sums [10], average Lyapunov functions [2, 3], and characteristic

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exponents for smooth dynamical systems [13, 14], homeomorphisms of metric spaces [7], and linear skew-product semiflows [5]. In each of these settings, it is the asymptotic behavior of these subadditive functions as $t \to \infty$ that is of interest. In his ground breaking work, Kingman [8] provided the first systematic study of the long-term behavior of subadditive functions from an ergodic point of view. Kingman's subadditive ergodic theorem [8] assures that $(1/t)\,F(x,t)$ has a well-defined limit almost surely for any ϕ-invariant measure. The purpose of this paper is to provide uniform upper bounds for the limiting values of $(1/t)\,F(x,t)$ in terms of these well-defined limits.

The paper is organized as follows. In Section 2, we introduce the main definitions: the maximal growth rate of a subadditive function and the growth rate of a subbadditive function with respect to an ergodic measure. The main theorem asserts that the maximal growth rate equals the supremum of the growth rates with respect to ergodic measures. In Section 3, we prove the main result. In Section 4, we derive three applications of the main result. First, we show that the maximal (respectively minimal) growth rate of the Birkhoff sums of a continuous function equals the supremum (respectively infimum) of the average of this function with respect to any ergodic measure. Second, we consider the dynamical spectrum and the measurable spectrum of a skew-product semiflow on Banach bundles. Sacker and Sell [16] defined the dynamical spectrum Σ_{dyn} for a finite-dimensional linear skew-product flow π over a suitable base space X to be the set of values $\lambda \in \mathbf{R}$ where the shifted semiflow π_λ fails to have an exponential dichotomy. This definition was extended to an infinite dimensional setting by Magalhães [9]. Alternatively, Johnson *et al.* [5] defined the measurable spectrum Σ_{meas} to be the closure of the characteristic exponents of π as determined by the multiplicative ergodic theorem [5, 13, 14]. The existence of the measurable spectrum in the infinite dimensional setting was proven by Ruelle [15] and Mañé [11]. In the spirit of Johnson *et al.* [5] we prove that

$$\partial\Sigma_{\mathrm{dyn}} \subseteq \Sigma_{\mathrm{meas}} \subseteq \Sigma_{\mathrm{dyn}}$$

thereby extending their result to the infinite dimensional setting. As our final application, the main result is applied to the study of average Lyapunov functions [2, 3] that are used to prove that certain positively invariant sets are repelling. They arise often in biological applications [4] and in these cases it is useful to know on what set it is necessary to check whether a candidate function is in fact an average Lyapunov function. We show that it is sufficient to check on the minimal center of attraction of the semiflow (a subset of the Birkhoff center of the semiflow).

2. DEFINITIONS AND STATEMENT OF MAIN RESULT

Given a semiflow, we restrict our attention to continuous functions

$$F\colon X \times \mathbf{T}_+ \to \mathbf{R}, \qquad F_t(x) = F(x, t)$$

that are *subadditive* with respect to ϕ (i.e., satisfy (1)). To study the measure-theoretic growth rates of these functions, let $\mathscr{M}_{\mathrm{inv}}(\phi)$ denote the space of Borel probability measures that are ϕ-invariant and let $\mathscr{M}_{\mathrm{erg}}(\phi) \subseteq \mathscr{M}_{\mathrm{inv}}(\phi)$ denote those invariant measures for which ϕ is ergodic. Given $\mu \in \mathscr{M}_{\mathrm{erg}}(\phi)$, Kingman's subadditive ergodic theorem [8] asserts that there exists a Borel set $U \subseteq X$ such that $\mu(U) = 1$ and

$$\lim_{t \to \infty} \frac{1}{t} F_t(x) = \inf_{t > 0} \frac{1}{t} \int_X F_t \, d\mu$$

for all $x \in U$. Hence, it makes sense to define the *growth rate of F_t with respect to μ* to be

$$\mathrm{GR}(F, \mu) = \inf_{t > 0} \frac{1}{t} \int_X F_t \, d\mu.$$

In general, however, the growth rate of F_t is not well defined at every point of X. Therefore, at best we can hope to find a uniform upper bound for the growth rate of a subadditive function. With this purpose in mind, we define the *maximal growth rate of F_t* to be

$$\mathrm{GR}^+(F) = \sup_{x \in X} \limsup_{t \to \infty} \frac{1}{t} F_t(x).$$

Our main result relates these measure-theoretic and dynamical definitions.

THEOREM 1. *Let $F\colon X \times \mathbf{T}_+ \to \mathbf{R}$ be a continuous subadditive function with respect to the semiflow ϕ. Then*

$$\begin{aligned} \mathrm{GR}^+(F) &= \sup\{\mathrm{GR}(F, \mu) : \mu \in \mathscr{M}_{\mathrm{erg}}(\phi)\} \\ &= \inf_{t > 0} \frac{1}{t} \sup_{x \in X} F_t(x). \end{aligned}$$

Theorem 1 shows for what invariant subset $K \subseteq X$ it is sufficient to evaluate the growth rate of F. This set is called the *minimal center of attraction* (see [10] or [12]) of ϕ, the unique compact positively invariant set $\mathscr{MC}(\phi)$ which satisfies two conditions:

(1) If U is any neighborhood of $\mathscr{MC}(\phi)$, let $\mathbf{1}_U$ be the characteristic function for U (i.e., $\mathbf{1}_U(x)=1$ if $x \in U$ else $\mathbf{1}_U(x)=0$). Then for all $x \in X$

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_U(\phi_s x)\, ds = 1$$

if $\mathbf{T}_+ = \mathbf{R}_+$ or

$$\lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \mathbf{1}_U(\phi_i x)\, ds = 1$$

if $\mathbf{T}_+ = \mathbf{Z}_+$.

(2) If $K \subseteq X$ is any other compact positively invariant set satisfying condition (1), then $\mathscr{MC}(\phi) \subseteq K$.

The Birkhoff ergodic theorem implies (see, e.g., Exercises I.8.3 and II.1.5 in [10]) that

$$\mathscr{MC}(\phi) = \overline{\bigcup_{\mu} \operatorname{supp}(\mu)},$$

where the union is taken over $\mu \in \mathscr{M}_{\mathrm{erg}}(\phi)$ and where $\operatorname{supp}(\mu)$ denotes the support of μ. Consequently, Theorem 1 implies

$$\mathrm{GR}^+(F) = \mathrm{GR}^+(F \mid \mathscr{MC}(\phi)).$$

It is worth noting that by the Poincaré recurrence theorem $\mathscr{MC}(\phi)$ is contained in the *Birkhoff center* of ϕ (i.e., the closure of the recurrent points). This inclusion can be proper. For instance, Nemystkii and Stepanov [12] provide an example of a flow on a two torus whose Birkhoff center is the entire torus but whose minimal center of attraction is a single point.

Theorem 1 can also be used to find a uniform lower bound on the growth rate of a *superadditive function* F with respect to ϕ: a continuous function F: $X \times \mathbf{T}_+ \to \mathbf{R}$ that satisfies

$$F(x, t+s) \geqslant F(x, t) + F(\phi_t x, s).$$

In this case, $-F(x, t)$ is subadditive. Hence, applying Theorem 1 to $-F$, we get that

$$\inf_{x \in X} \liminf_{t \to \infty} \frac{1}{t} F_t(x) = \inf\{\mathrm{GR}(F_t, \mu) : \mu \in \mathscr{M}_{\mathrm{erg}}(\phi)\} = \sup_{t>0} \frac{1}{t} \inf_{x \in X} F_t(x).$$

We shall denote these equivalent quantities $\mathrm{GR}^-(F)$, the *minimal growth rate* of the superadditive function F.

3. PROOF OF THEOREM 1

To avoid confusion, throughout this section we let t denote an element of $\mathbf{R}_+$ and n denote an element of $\mathbf{Z}_+$. We begin by assuming that $\mathbf{T}_+ = \mathbf{Z}_+$ and by proving

$$\mathrm{GR}^+(F) \geqslant \sup\{\mathrm{GR}(F,\mu) : \mu \in \mathscr{M}_{\mathrm{erg}}(\phi)\} \geqslant \inf_{n\geqslant 1} \sup_{x\in X} \frac{1}{n} F_n(x). \tag{2}$$

Kingman's subadditive ergodic theorem implies the first inequality in (2). To prove the second inequality in (2), choose $\varepsilon > 0$. We will show that there exists a $\mu \in \mathscr{M}_{\mathrm{erg}}(\phi)$ such that

$$\mathrm{GR}(F,\mu) + 2\varepsilon \geqslant \inf_{n\geqslant 1} \sup_{x\in X} \frac{1}{n} F_n(x). \tag{3}$$

To prove (3), for every $n \in \mathbf{Z}_+$ choose $y_n \in X$ such that

$$\frac{1}{n} F_n(y_n) + \varepsilon \geqslant \inf_{n\geqslant 1} \sup_{x\in X} \frac{1}{n} F_n(x). \tag{4}$$

Define a sequence of Borel probability measures η_n on X by

$$\eta_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\phi_i y_n},$$

where δ_x is the Dirac measure concentrated at the point x. Compactness of X implies there exists a subsequence of measures η_{n_k} that converges to a measure v in the weak* topology. For notational convenience, we write $v_k = \eta_{n_k}$ and $x_k = y_{n_k}$. To show that v is ϕ-invariant, let $f: X \to \mathbf{R}$ be a continuous function. Weak* convergence of the v_k implies

$$\begin{aligned}
\int_X f(\phi_1 x)\, dv(x) &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} f(\phi_{i+1} x_k) \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} f(\phi_i x_k) + \lim_{k\to\infty} \frac{1}{n_k} (f(\phi_{n_k} x_k) - f(x_k)) \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} f(\phi_i x_k) \\
&= \int_X f(x)\, dv(x).
\end{aligned}$$

Since f was an arbitrary continuous function, v is ϕ-invariant.

Next we prove a lemma based on estimates found in Katznelson and Weiss' proof of the subadditive ergodic theorem [6].

LEMMA 1. *Let n_k be a strictly increasing sequence of positive integers. Then*

$$\limsup_{k\to\infty} \frac{1}{mn_k} \sum_{i=0}^{n_k-1} F_m(\phi_i x) \geqslant \limsup_{k\to\infty} \frac{1}{n_k} F_{n_k}(x)$$

for any $x \in X$ and $0 \neq m \in \mathbf{Z}_+$.

Proof. Assume $n_k > m$. For each i between 1 and m there exists a unique choice of integers $c(i,k) \geqslant 0$ and $0 \leqslant r(i,k) \leqslant m$ such that $n_k = i + c(i,k)\,m + r(i,k)$. By subadditivity,

$$\begin{aligned} F_{n_k}(x) &\leqslant F_i(x) + F_{c(i,k)\,m}(\phi_i x) + F_{r(i,k)}(\phi_{i+c(i,k)\,m} x) \\ &\leqslant F_i(x) + \sum_{j=0}^{c(i,k)-1} F_m(\phi_{i+jm} x) + F_{r(i,k)}(\phi_{i+c(i,k)\,m} x). \end{aligned}$$

Summing both sides over i from 1 to m, we get

$$\begin{aligned} mF_{n_k}(x) &\leqslant \sum_{i=1}^{m} F_i(x) + \sum_{i=1}^{m} \sum_{j=0}^{c(i,k)-1} F_m(\phi_{i+jm} x) + \sum_{i=1}^{m} F_{r(i,k)}(\phi_{i+c(i,k)\,m} x) \\ &= \sum_{i=1}^{m} F_i(x) + \sum_{i=1}^{n_k-m} F_m(\phi_i x) + \sum_{i=1}^{m} F_{r(i,k)}(\phi_{i+c(i,k)\,m} x), \end{aligned}$$

where the second line follows from the definition of $c(i,k)$ and $r(i,k)$. Dividing both sides by mn_k and rearranging terms, we get

$$\begin{aligned} \frac{1}{n_k} F_{n_k}(x) \leqslant \frac{1}{n_k m} \Bigg(&\sum_{i=1}^{m} F_i(x) + \sum_{i=0}^{n_k-1} F_m(\phi_i x) - F_m(x) \\ &- \sum_{i=n_k-m+1}^{n_k-1} F_m(\phi_i x) + \sum_{i=1}^{m} F_{r(i,k)}(\phi_{i+c(i,k)\,m} x) \Bigg). \end{aligned}$$

As $0 \leqslant r(i,k) \leqslant m$ and $0 \leqslant n_k - c(m,k)\,m \leqslant 2m$, continuity of F and compactness of X imply that the limit

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{n_k m} \Bigg(&\sum_{i=1}^{m} F_i(x) - F_m(x) - \sum_{i=n_k-m+1}^{n_k-1} F_m(\phi_i x) \\ &+ \sum_{i=1}^{m} F_{r(i,k)}(\phi_{i+c(i,k)\,m} x) \Bigg) \end{aligned}$$

exists and equals zero. Thus, taking the lim sup on both sides of (5) completes the proof of the lemma. ∎

Returning to the proof of Theorem 1, recall that $x_k = y_{n_k}$. Lemma 1 and (4) imply that

$$\inf_{m \geqslant 1} \frac{1}{m} \int_X F_m \, d\nu = \inf_{m \geqslant 1} \lim_{k \to \infty} \frac{1}{mn_k} \sum_{i=0}^{n_k - 1} F_m(\phi_i x_k)$$

$$\geqslant \limsup_{k \to \infty} \frac{1}{n_k} F_{n_k}(x_k)$$

$$\geqslant \inf_{n \geqslant 1} \sup_{x \in X} \frac{1}{n} F_n(x) - \varepsilon.$$

The ergodic decomposition theorem (see [10, Chap. II, Theorem 6.4], implies

$$\int_X F_m \, d\nu = \int_X \left(\int_X F_m \, d\nu_x \right) d\nu,$$

where ν_x are Borel probability measures for which ϕ is ergodic. It follows that there exists an ergodic measure $\mu = \nu_x$ for some $x \in X$ such that (3) holds. Taking the limit as $\epsilon \to 0$ completes the proof of the second inequality in (2).

To complete the proof in the discrete case, we need the following well-known lemma (see, for example, [1, p. 28]).

LEMMA 2. *If a_n is a sequence of real numbers such that $a_{n+m} \leqslant a_n + a_m$ for all $n, m \in \mathbf{Z}_+$, then*

$$\lim_{n \to \infty} \frac{1}{n} a_n = \inf_{n \geqslant 1} \frac{1}{n} a_n.$$

Similarly if $a \colon \mathbf{R}_+ \to \mathbf{R}$ is a continuous function such that $a(t+s) \leqslant a(t) + a(s)$ for all $s, t \in \mathbf{R}_+$, then

$$\lim_{t \to \infty} \frac{1}{t} a_t = \inf_{t > 0} \frac{1}{t} a_t.$$

Subadditivity of F implies that the sequence $a_n = \sup_{x \in X} F_n(x)$ satisfies the conditions of Lemma 2. Therefore

$$\inf_{n \geqslant 1} \sup_{x \in X} \frac{1}{n} F_n(x) = \lim_{n \to \infty} \frac{1}{n} \sup_{x \in X} F_n(x) \geqslant \mathrm{GR}^+(F_n)$$

which completes the proof of Theorem 1 in the discrete case.

Now consider a continuous-time semiflow $\phi(t, x)$ and a subadditive function $F(x, t)$ where $\mathbf{T}_+ = \mathbf{R}_+$. Define $\phi^1(x, n) = \phi(x, n)$ to be the time-one map for ϕ. With ϕ^1 we associate the subadditive function $F^1(x, n) = F(x, n)$. The proof of Theorem 1 in the continuous- time case follows from the discrete-time case and the next lemma.

LEMMA 3.

$$\mathrm{GR}^+(F^1) = \mathrm{GR}^+(F) \tag{6}$$

$$\inf_{n \geqslant 1} \sup_{x \in X} \frac{1}{n} F_n^1(x) = \inf_{t > 0} \sup_{x \in X} \frac{1}{t} F_t(x) \tag{7}$$

$$\sup\{\mathrm{GR}(F^1, \mu) : \mu \in \mathcal{M}_{\mathrm{erg}}(\phi^1)\} = \sup\{\mathrm{GR}(F, \mu) : \mu \in \mathcal{M}_{\mathrm{erg}}(\phi)\}. \tag{8}$$

Proof. Given $t \in \mathbf{R}$ let $[t]$ denote its integer part. Continuity of F and compactness of X implies there exists $K > 0$ such that $|F(x, t)| \leqslant K$ for all $x \in X$ and $0 \leqslant t \leqslant 1$. Given $t > 1$, subadditivity of F implies that

$$F(x, t) - F(x, [t]) \leqslant F(\phi_{[t]} x, t - [t]) \leqslant K. \tag{9}$$

Since $\lim_{t \to \infty}(t/[t]) = 1$, (9) implies

$$\limsup_{t \to \infty} \frac{1}{t} F(x, t) \leqslant \limsup_{t \to \infty} \frac{1}{t} F(x, [t]) = \limsup_{n \to \infty} \frac{1}{n} F^1(x, n).$$

Alternatively, since $\mathbf{Z}_+ \subset \mathbf{R}_+$, the opposite inequality holds and we have

$$\limsup_{t \to \infty} \frac{1}{t} F(x, t) = \limsup_{n \to \infty} \frac{1}{n} F(x, n). \tag{10}$$

Equation (10) implies (6).

To prove (7), set $a_t = \sup_{x \in X} F_t(x)$. Lemma 2 implies that

$$\lim_{n \to \infty} \frac{1}{n} a_n = \inf_{n \geqslant 1} \frac{1}{n} a_n, \qquad \lim_{t \to \infty} \frac{1}{t} a_t = \inf_{t > 0} \frac{1}{t} a_t.$$

Since $\mathbf{Z}_+ \subset \mathbf{R}_+$, it follows that $\lim_{n \to \infty}(1/n)\, a_n = \lim_{t \to \infty}(1/t)\, a_t$ which completes the proof of (7).

To prove (8), notice that the characterization of $\mathcal{MC}(\phi^1)$ and $\mathcal{MC}(\phi)$ as the closure of the supports of the ergodic measures implies that

$$\sup\{\mathrm{GR}(F^1,\mu):\mu\in\mathcal{M}_{\mathrm{erg}}(\phi^1)\}=\mathrm{GR}^+(F^1\mid\mathcal{MC}(\phi^1))$$
$$\sup\{\mathrm{GR}(F,\mu):\mu\in\mathcal{M}_{\mathrm{erg}}(\phi)\}=\mathrm{GR}^+(F\mid\mathcal{MC}(\phi)).$$

Therefore by (10) it is sufficient to show that $\mathcal{MC}(\phi^1)=\mathcal{MC}(\phi)$. Since any ergodic measure μ for ϕ is an ergodic measure for ϕ^1, it follows that $\mathcal{MC}(\phi)\subseteq\mathcal{MC}(\phi^1)$. The inclusion in the opposite direction follows from the definition of the minimal center of attraction. ∎

4. APPLICATIONS

4.1. *Birkhoff Sums*

An immediate application of Theorem 1 is to Birkhoff sums (see Exercise I.8.5 in [10]). We state the result in the case when $\mathbf{T}_+=\mathbf{R}_+$. An analogous statement holds for the discrete-time case.

COROLLARY 1. *Let ϕ be a continuous-time semiflow on a compact metric space X. If $f: X\to\mathbf{R}$ is a continuous function then*

$$\sup_{x\in X}\ \limsup_{t\to\infty}\frac{1}{t}\int_0^t f(\phi_s x)\,ds=\sup\left\{\int_X f\,d\mu:\mu\in\mathcal{M}_{\mathrm{erg}}(\phi)\right\}$$

$$\inf_{x\in X}\ \liminf_{t\to\infty}\frac{1}{t}\int_0^t f(\phi_s x)\,ds=\inf\left\{\int_X f\,d\mu:\mu\in\mathcal{M}_{\mathrm{erg}}(\phi)\right\}.$$

Proof. Define $F(x,t)=\int_0^t f(\phi_s x)\,ds$. F is continuous and additive (i.e., superadditive and subadditive). Theorem 1 implies that $\mathrm{GR}^+(F)=\sup_\mu \mathrm{GR}(F,\mu)$ and $\mathrm{GR}^-(F)=\inf_\mu \mathrm{GR}(F,\mu)$. The proof of the corollary is completed by observing that the Birkhoff ergodic theorem implies that $\mathrm{GR}(F,\mu)=\int_X f\,d\mu$.

4.2. *Spectra for Linear Skew-Product Semiflows on Banach Bundles*

In this section we assume that $\phi: X\times\mathbf{R}\to X$ is a continuous-time flow on a compact metric space X. Following the work of Sacker and Sell [16], Johnson *et al.* [5], and Magalhães [9], we study the spectral properties of linear-skew product semiflows on Banach vector bundles over ϕ. A *Banach vector bundle* E over X is a triad $(E, p, \|\cdot\|)$ where E is a topological space, $p: E\to X$ a continuous map (called the canonical projection), $\|\cdot\|: E\to\mathbf{R}$ a continuous function and for every $x\in X$ the set $E(x)=p^{-1}(x)$ is endowed with a vector space structure such that $\|\cdot\|_x: E(x)\to\mathbf{R}$ is a Banach norm on $E(x)$ and the topology induced by this norm coincides with the relative topology of $E(x)$. $E(x)$ is called the *fibre* over x. Points in E can be represented as ordered pairs (x, v) where $x \in X$ and $v \in E(x)$. A semiflow $\pi: E \times \mathbf{R}_+ \to E$ is said to be a *linear skew-product semiflow* on E if

$$\pi(x, v, t) = (\phi(x, t), \Phi(x, t)\, v),$$

where $\Phi(x, t)$ is a bounded linear map that sends the fiber $E(x)$ to the fiber $E(\phi_t x)$. Since π is only defined for $t \geqslant 0$, it is useful to identify the points (x, v) in E through which there is a *backward continuation* of π. To this end, we define the set

$$B = \{(x, v) \in E : \text{there is exactly one continuous function } (u, w) : (-\infty, 0] \to E \text{ such that } u(0) = x,\ w(0) = v \text{ and } \pi(u(s), w(s), t) = (u(t+s), w(t+s)) \text{ for all } s \leqslant 0 \text{ and all } t \in [0, -s]\}.$$

For any point $(x, v) \in B$ and $t \leqslant 0$ we set $\Phi(x, t)\, v$ equal to $(u(t), w(t))$ where (u, w) is the unique backward continuation of (x, v). We define the *stable set* of π by

$$S = \{(x, v) \in E : \lim_{t \to \infty} |\Phi(x, t)\, v| = 0\}$$

and the *unstable set* of π by

$$U = \{(x, v) \in B : \lim_{t \to -\infty} |\Phi(x, t)\, v| = 0\}.$$

Notice that the set S is positively invariant under π (i.e., $\pi_t S \subseteq S$ for all $t \geqslant 0$) and the set U is invariant under π (i.e., $U \subseteq B$ and $\pi_t U \subseteq U$ for all $t \in \mathbf{R}$). It is easy to check that S and U are vector sub-bundles of E.

The linear skew-product flow π is said to have an *exponential dichotomy* provided that there exists a continuous family of linear projectors $P(x)$ of the fibers $E(x)$ and constants $K, \alpha > 0$ such that

(i) $N = \{(x, v) \in E : P(x)\, v = 0\} \subseteq B$.

(ii) $\|\Phi(x, t)\, P(x)\| \leqslant K e^{-\alpha t}$ for all $t \geqslant 0$ and $x \in X$.

(iii) $\|\Phi(x, t)(I - P(x))\| \leqslant K e^{\alpha t}$ for all $t \leqslant 0$ and $x \in X$.

Notice that (i) implies that (iii) makes sense. Whenever π admits an exponential dichotomy, the unstable set U equals N and the stable set S equals $\{(x, P(x)\, v) : (x, v) \in E\}$. Consequently, $E = S \oplus U$ where $\oplus$ denotes a Whitney sum.

Given $\lambda \in \mathbf{R}$, define the linear skew-product semiflow π_λ by

$$\pi_\lambda(x, v, t) = (\phi_t x, e^{-\lambda t}\Phi(x, t)\, v)$$

for $t \geqslant 0$ and $(x, v) \in E$. The *resolvent* $\rho(E, \pi)$ of π is defined to be the set

$$\rho(E, \pi) = \{\lambda \in \mathbf{R}_+ : \pi_\lambda \text{ admits an exponential dichotomy}\}.$$

Given $\lambda \in \rho(E, \pi)$, let U_λ and S_λ denote the unstable and stable sets of π_λ. The *dynamical spectrum* of π is defined to be the set

$$\Sigma_{\text{dyn}}(E, \pi) = \mathbf{R} \backslash \rho(E, \pi).$$

Magalhães [9, Theorem 2.1] provided the following characterization of the dynamical spectrum.

THEOREM 2 (Magalhães, 1987). *Let $\pi = (\phi, \Phi)$ be a linear skew-product semiflow over a compact connected metric space X. Assume that $\Phi(x, t)$ is a compact linear operator for all $t \geqslant 0$ and $x \in X$. Then the dynamical spectrum $\Sigma_{\text{dyn}}(E, \pi)$ is closed, bounded above, and equals the union of closed intervals. These intervals are called the* spectral intervals *and in this setting an interval $[a, b]$ is allowed to degenerate to a point when $a = b$.*

Associated with each spectral interval there is a spectral bundle *V of E which satisfies the following properties*:

(1) *If $\mu, \lambda \in \rho(E, \pi)$ and $(\mu, \lambda) \cap \Sigma_{dyn}(E, \pi) = [a, b]$ then the spectral bundle V associated with $[a, b]$ has finite dimension, satisfies $V = U_\mu \cap S_\lambda$ and is invariant under π.*

(2) *If $\lambda \in \rho(E, \pi)$ and $(-\infty, \lambda) \cap \Sigma_{dyn}(E, \pi) = (-\infty, b]$, then the spectral bundle V associated with $(-\infty, b]$ satisfies $V = S_\lambda$ and is positively invariant under π.*

(3) *If $\lambda \in \rho(E, \pi)$ then the number of spectral intervals included in (λ, ∞) is finit*e.

Remarks. Magalhães original statement of the theorem assumed that X is a compact connected smooth Banach manifold. It is easily seen that his proof holds for compact connected metric spaces.

To define the measurable counterpart to $\Sigma_{\text{dyn}}(E, \pi)$, we first state a theorem of Mañé [11] which is the infinite dimensional counterpart to Oseledec's multiplicative ergodic theorem [13]. Ruelle [15] proved a similar theorem for Hilbert space vector bundles.

THEOREM 3 (Mañé, 1983). *Let $\pi = (\phi, \Phi)$ be a linear skew-product semiflow over a compact metric space X. Assume that $\Phi(x, t)$ is compact and* *injective for all* $t \geqslant 0$ *and* $x \in X$. *Then there is a Borel set* Γ *such that* $\mu(\Gamma) = 1$ *for all* $\mu \in \mathscr{M}_{inv}(\phi)$ *and such that every* $x \in \Gamma$ *satisfies one of the following three conditions*

(1) $\lim_{t \to \infty} (1/t) \ln \|\Phi(x, t)\| = -\infty$.

(2) *There exists a* $k(x) \in \mathbf{Z}_+$ *and a splitting*

$$E(x) = E_1(x) \oplus \cdots \oplus E_{k(x)}(x) \oplus F_\infty(x)$$

and numbers $\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$ *such that*

(a) $E_i(x)$ *is finite dimensional for all* $1 \leqslant i \leqslant k(x)$.

(b) $\lim_{t \to \pm\infty} (1/t) \ln \|\Phi(x, t)\, v\| = \lambda_i(x)$ *for all* $0 \neq v \in E^i(x)$ *and* $1 \leqslant i \leqslant k(x)$.

(c) $\lim_{t \to \infty} (1/t) \ln \|\Phi(x, t) \mid F_\infty(x)\| = -\infty$.

(3) *There exist subspaces* $E_i(x)$, $F_i(x)$, $i = 1, 2, ...,$ *and real numbers* $\lambda_1(x) > \lambda_2(x) > \cdots$ *such that*:

(a) $E_i(x)$ *is finite dimensional for all* $i \geqslant 1$.

(b) $\lim_{n \to \infty} \lambda_n(x) = -\infty$.

(c) $E_1(x) \oplus \cdots \oplus E_i(x) \oplus F_i(x) = E(x)$ *for all* i.

(d) *For all* i *and* $0 \neq v \in E_i(x)$,

$$\lim_{t \to \pm\infty} \frac{1}{t} \ln \|\Phi(x, t)\, v\| = \lambda_i(x).$$

(e) *For all* i,

$$\lim_{t \to \infty} \frac{1}{t} \ln \|\Phi(x, t) \mid F^i(x)\| = \lambda_{i+1}(x).$$

Following Johnson *et al.* [5] we define the *measurable spectrum* of a linear skew-product semiflow π by

$$\Sigma_{\text{meas}}(E, \pi) = \overline{\{\lambda_i(x) : x \in \Gamma, 1 \leqslant i \leqslant k(x)\}},$$

where the $\lambda_i(x)$ are the *characteristic exponents* as defined in Mañé's theorem and where we set $k(x) = 0$ or ∞ when $x \in \Gamma$ corresponds to a point in case (1) or (3) of Theorem 3.

Using Theorem 1 in conjunction with the results of Magalhães and Mañé, we get the following result.

THEOREM 4. *Let $\pi = (\phi, \Phi)$ be a linear skew-product semiflow over a compact connected metric space X. Assume that $\Phi(t, x)$ is a compact and injective operator for all $x \in X$ and $t \geqslant 0$. Then*

$$\partial \Sigma_{\text{dyn}}(E, \pi) \subseteq \Sigma_{\text{meas}}(E, \pi) \subseteq \Sigma_{\text{dyn}}(E, \pi),$$

where $\partial \Sigma_{\text{dyn}}(E, \pi)$ denotes the boundary of $\Sigma_{\text{dyn}}(E, \pi)$.

Proof. We first show that

$$\Sigma_{\text{meas}}(E, \pi) \subseteq \Sigma_{\text{dyn}}(E, \pi).$$

To this end, let $\lambda \in \mathbf{R}$ and $(y, w) \in E$ be such that

$$\lim_{t \to \infty} \frac{1}{t} \ln |\Phi(y, t)\, w| = \lambda. \tag{11}$$

We want to show that $\lambda \in \Sigma_{\text{dyn}}(E, \pi)$. Arguing negatively, suppose that $\lambda \in \rho(E, \pi)$. Then π_λ admits an exponential dichotomy $E = S_\lambda \oplus U_\lambda$ and there exist $K, \beta > 0$ such that

$$|\Phi(x, t)\, v_s| \leqslant K e^{(\lambda - \beta)\, t}\, |v_s| \qquad \text{for all} \quad x \in X, \quad v_s \in S_\lambda(x), \quad t \geqslant 0 \tag{12}$$

$$|\Phi(x, -t)\, v_u| \leqslant K e^{-(\beta + \lambda)\, t}\, |v_u| \qquad \text{for all} \quad x \in X, \quad v_u \in U_\lambda(x), \quad t \geqslant 0. \tag{13}$$

In this case, we can write $w = w_s + w_u$ for some $w_s \in S_\lambda(y)$ and $w_u \in U_\lambda(y)$. Relations (11) and (12) imply that w_u cannot equal zero. Relation (13) implies that for $t \geqslant 0$

$$\begin{aligned} |w_u| &= |\Phi(\phi_t y, -t)\, \Phi(y, t)\, w_u| \\ &\leqslant K e^{-(\beta + \lambda)\, t}\, |\Phi(y, t)\, w_u|. \end{aligned}$$

Hence, we get that for all $t \geqslant 0$

$$|\Phi(y, t)\, w_u| \geqslant \frac{1}{K} e^{(\lambda + \beta)\, t}\, |w_u|$$

which contradicts (11).

Next we prove the inclusion,

$$\partial \Sigma_{\text{dyn}}(E, \pi) \subseteq \Sigma_{\text{meas}}(E, \pi).$$

To this end, let I be a spectral interval of the form $[a, b]$ or $(-\infty, b]$ and let V be its associated spectral bundle as given by Theorem 2. Define

$$F(x, t) = \ln \|\Phi(x, t) \mid V(x)\|$$

for all $x \in X$ and $t \geqslant 0$. F is continuous and subadditive with respect to ϕ. For any $\mu \in \mathscr{M}_{\text{erg}}(\phi)$, $\text{GR}(F, \mu)$ is the maximal Lyapunov exponent of $\pi \mid V$ with respect to μ (see, e.g., [11] or [15]). In particular there exists $(y, w) \in V$ such that

$$\lim_{t \to \infty} \frac{1}{t} \ln |\Phi(y, t)\, w| = \text{GR}(F, \mu).$$

Our arguments in the previous paragraph applied to $\pi \mid V$ imply that $\text{GR}(F, \mu) \in \Sigma_{\text{dyn}}(V, \pi) = I$. On the other hand, Theorem 1 implies that $\text{GR}^+(F) = \sup\{\text{GR}(F, \mu) : \mu \in \mathscr{M}_{\text{erg}}(\phi)\}$. Hence $\text{GR}^+(F) \in I$. We claim that $\text{GR}^+(F) = b$. Arguing negatively assume that $\text{GR}^+(F) < b$ then Theorem 1 implies that there exists $\varepsilon > 0$ such that

$$\inf_{t > 0} \frac{1}{t} \sup_{x \in X} F(x, t) < b - \varepsilon.$$

Therefore there exists $T > 0$ such that

$$e^{-bT} \|\Phi(x, T) \mid V(x)\| \leqslant e^{-\varepsilon T}$$

for all $x \in X$. Submultiplicativity of linear operators with respect to the operator norm implies that for all $n \in \mathbf{Z}_+$, $x \in X$,

$$e^{-bnT} \|\Phi(x, nT) \mid V(x)\| \leqslant e^{-\varepsilon nT}. \tag{14}$$

Let

$$K = \sup_{x \in X,\, 0 \leqslant t \leqslant T} e^{T} \|\Phi(x, t) \mid V(x)\|.$$

Given any $t \geqslant 0$, there is a unique nonnegative integer n and real $0 \leqslant r \leqslant T$ such that $t = nT + r$. This observation, inequality (14) and our choice of K imply

$$e^{-bt} \|\Phi(x, t) \mid V(x)\| \leqslant K e^{-\varepsilon t}.$$

This inequality implies that $\pi_b \mid V$ admits an exponential dichotomy with projectors $P(x)$ equal to the identity map. Hence $b \in \rho(V, \pi)$ contradicting our choice of b. Therefore $\mathrm{GR}^+(F)=b$ and by Theorem 1, $b\in\Sigma_{\mathrm{meas}}(V,\pi)\subseteq\Sigma_{\mathrm{meas}}(E,\pi)$. When the spectral interval I is of the form $[a,b]$, we claim that $a\in\Sigma_{\mathrm{meas}}(E,\pi)$. To prove this claim, we define the superadditive function

$$G(x,t)=-\ln\|\Phi(x,-t)\mid V(x)\|$$

for $x\in X$ and $t\geqslant 0$. In this case, for any $\mu\in\mathcal{M}_{\mathrm{erg}}(\mu)$, $\mathrm{GR}(G,\mu)$ is the minimal Lyapunov exponent of $\pi\mid V$. Mimicking our previous arguments for b, it follows that $a=\mathrm{GR}^-(G)=\inf\{\mathrm{GR}(G,\mu):\mu\in\mathcal{M}_{\mathrm{erg}}(\phi)\}$. Hence $a\in\Sigma_{\mathrm{meas}}(E,\pi)$. ∎

Remarks. Johnson *et al.* [5, Theorem 2.3] proved Theorem 4 for linear skew-product flows on finite-dimensional spaces. A related theorem was proven in the discrete case by the author [17].

4.3. *Average Lyapunov Functions*

Consider a semiflow ϕ: $Y\times\mathbf{T}_+\to Y$ on a locally compact metric space Y. Assume X is a compact subset of Y with empty interior such that X and $Y\backslash X$ are positively invariant. Motivated by applications, various methods have been developed to determine whether X is a *uniform repellor*, i.e., there exists $\eta>0$ such that for all $y\in Y\backslash X$, $\liminf_{t\to\infty} d(\phi_t y, X)>\eta$ (see, for example, [4]). One of these methods uses what is commonly referred to as an average Lyapunov function [2, 3]: Given $U\subseteq Y$ an open neighborhood of X and a continuous function P: $U\to\mathbf{R}_+$, define F: $X\times\mathbf{T}_+\to\mathbf{R}$ by

$$F(x,t)=\ln \liminf_{y\to x,\ y\in U\backslash X}\frac{P(\phi_t y)}{P(y)}. \tag{15}$$

P is called an *average Lyapunov function* provided that $P^{-1}(0)=X$ and

$$\sup_{t>0} F(x,t)>0$$

for all $x\in X$.

Remarks. Recall P is Lyapunov function if $P(\phi_t y)>P(y)$ for all $y\in U\backslash X$ and $t>0$. Not all Lyapunov functions are average Lyapunov functions. For example, consider $\dot{x}=x(1-x^2)$ with $P(x)=x$. However, the advantage of an average Lyapunov function is that it gives a condition that only needs be checked at X.

THEOREM (Hutson, 1984). *Let Y, X, and ϕ be as defined above. If there exists an average Lyapunov function for X, then X is uniformly repelling.*

As F is superadditive, Theorem 1 immediately implies the following corollary.

COROLLARY 2. *Let U, X, Y, and ϕ be as defined above. Let P: $U \to \mathbf{R}_+$ be a continuous function such that $P^{-1}(0) = X$. If $F(x, t)$ as defined in* (15) *is continuous and*

$$\inf \left\{ \liminf_{t \to \infty} \frac{1}{t} F(x, t) : x \in \mathscr{M}(\phi \mid X) \right\} > 0,$$

then X is uniformly repelling.

Remarks. Hutson [2, Corollary 2.3] proved that it suffices to check that $\liminf_{t \to \infty} (1/t)\, F(x, t) > 0$ for $x \in L^+(\phi \mid X) = \overline{\bigcup_{x \in X} \omega(x)}$ where $\omega(x)$ denotes the ω-limit set of the point x. Since $L^+(\phi \mid X)$ contains the Birkhoff center of $\phi \mid X$, Corollary 3 improves this result.

Under additional assumptions (i.e., ϕ is a dissipative Lipschitz flow on $Y = \mathbf{R}^n_+$ and $X = \partial \mathbf{R}^n_+$), Hutson [3, Theorem 5.2] proved that if X is uniformly repelling then there exists an average Lyapunov function P such that F as defined by (15) is continuous. Hence Corollary 3 can be interpreted as saying that behavior of ϕ near $\mathscr{M}(\phi \mid X)$ determines whether X is a uniform repellor.

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REFERENCES

1. R. Bowen, "Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms," Lecture Notes in Mathematics, Vol. 470, Springer-Verlag, Berlin, 1975.
2. V. Hutson, A theorem on average Liapunov functions, *Monatsh. Math.* **98** (1984), 267–275.
3. V. Hutson, The stability under perturbations of repulsive sets, *J. Differential Equations* **76** (1988), 77–90.
4. V. Hutson and K. Schmitt, Permanence and the dynamics of biological systems, *Math. Biosci.* **111** (1992), 1–71.
5. R. A. Johnson, K. J. Palmer, and G. R. Sell, Ergodic properties of linear dynamical systems, *SIAM J. Math. Anal.* **18** (1987), 1–33.
6. Y. Katznelson and B. Weiss, A simple proof of some ergodic theorems, *Israel J. Math.* **42** (1982), 291–296.
7. Y. Kifer, Characteristic exponents of dynamical systems in metric spaces, *Ergodic Theory Dynamical Systems* **3** (1983), 119–127.

8. J. F. C. Kingman, Subadditive ergodic theory, *Ann. Probab.* **1** (1973), 883–909.
9. L. T. Magalhães, Persistence and smoothness of hyperbolic invariant manifolds for functional differential equations, *SIAM J. Math. Anal.* **18** (1987), 670–693.
10. R. Mañé, "Ergodic Theory and Differentiable Dynamics," Springer-Verlag, New York, 1983.
11. R. Mañé, Lyapunov exponents and stable manifolds for compact transformations, *in* "Geometric Dynamics" (J. Palis, Ed.), Lecture Notes in Mathematics, Vol. 1007, pp. 522–577, Springer-Verlag, Berlin, 1983.
12. V. V. Nemytskii and V. V. Stepanov, "Qualitative Theory of Differential Equations," Princeton Univ. Press, Princeton, 1960.
13. V. I. Oseledec, A multiplicative ergodic theorem, *Trans. Moscow Math. Soc.* **19** (1968), 197–231.
14. D. Ruelle, Ergodic theory of differentiable dynamical systems, *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 27–58.
15. D. Ruelle, Characteristic exponenets and invariant manifolds in Hilbert space, *Ann. Math.* **115** (1982), 243–290.
16. R. J. Sacker and G. R. Sell, A spectral theory for linear differential systems, *J. Differential Equations* **27** (1978), 320–358.
17. S. J. Schreiber, Expansion rates and Lyapunov exponents, *Discrete Conts. Dynam. Sys.* **3** (1997), 433–438.