

A Dynamical Trichotomy for Structured Populations Experiencing Positive Density-Dependence in Stochastic Environments

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Abstract Positive density-dependence occurs when individuals experience increased survivorship, growth, or reproduction with increased population densities. Mechanisms leading to these positive relationships include mate limitation, saturating predation risk, and cooperative breeding and foraging. Individuals within these populations may differ in age, size, or geographic location and thereby structure these populations. Here, I study structured population models accounting for positive density-dependence and environmental stochasticity i.e. random fluctuations in the demographic rates of the population. Under an accessibility assumption (roughly, stochastic fluctuations can lead to populations getting small and large), these models are shown to exhibit a dynamical trichotomy: (i) for all initial conditions, the population goes asymptotically extinct with probability one, (ii) for all positive initial conditions, the population persists and asymptotically exhibits unbounded growth, and (iii) for all positive initial conditions, there is a positive probability of asymptotic extinction and a complementary positive probability of unbounded growth. The main results are illustrated with applications to spatially structured populations with an Allee effect and age-structured populations experiencing mate limitation.

Keywords Structured populations · Environmental stochasticity · Allee effects · Positive density-dependence

1 Introduction

Higher population densities can increase the chance of mating success, reduce the risk of predation, and increase the frequency of cooperative behavior [5]. Hence, survivorship, growth, and reproductive rates of individuals can exhibit a positive

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relationship with density i.e. positive density-dependence. In single species models, positive density-dependence can lead to an Allee effect: the existence of a critical density below which the population tends toward extinction and above which the population persists [7, 12, 18, 22, 23]. Consequently, the importance of Allee effects have been widely recognized for conservation of at risk populations and the management of invasive species [5]. Populations experiencing environmental stochasticity and a strong Allee effect are widely believed to be especially vulnerable to extinction as the fluctuations may drive their densities below the critical threshold [6]. When population densities lie above the critical threshold for the unperturbed system, analyses and simulations of stochastic models support this conclusion [2, 7–9, 17, 21]. However, these studies also show that when population densities lie below the critical threshold, stochastic fluctuations can rescue the population from the deterministic vortex of extinction.

Individuals within populations often differ in diversity of attributes including age, size, gender, and geographic location [4]. Positive density-dependence may differentially impact individuals in populations structured by these attributes [5, 11]. This positive density-dependence can lead to an Allee threshold surface (usually a co-dimension one stable manifold of an unstable equilibrium) that separates population states that lead to extinction from those that lead to persistence [24].

While several studies have examined how environmental stochasticity and population structure interact to influence persistence of populations experiencing negative-density dependence [3, 13, 14, 20], I know of no studies that examine this issue for populations experiencing positive density-dependence. To address this gap, this paper examines stochastic, single species models of the form

$$X_{t+1} = A(X_t, \xi_{t+1})X_t \quad (1)$$

where $X_t = (X_{1,t}, X_{2,t}, \dots, X_{n,t}) \in [0, \infty)^n$ is a column-vector of population densities, $A(X_t, \xi_{t+1})$ is a $n \times n$ non-negative matrix that determines the population densities in the next year as a function of the current densities X_t and the environmental state ξ_{t+1} over the time interval $[t, t + 1)$. To focus on the effects of positive density-dependence, I assume that the entries of A are non-decreasing functions of the population densities. Under additional suitable assumptions described in Sects. 2 and 3, this paper shows that there is a dynamical trichotomy for (1): (i) asymptotic extinction occurs with probability one for all initial conditions, (ii) long-term persistence occurs with probability one for all positive initial conditions, or (iii) long-term persistence and asymptotic extinction occur with complementary positive probabilities for all positive initial conditions. The model assumptions and definitions are presented in Sect. 2. Exemplar models of spatially-structured populations and age-structured populations are also presented in this section. The main results and applications to the exemplar models appear in Sects. 3 and 4. Proofs of the main results are relegated to Sect. 5.

2 Models, Assumptions, and Definitions

Throughout this paper, I consider stochastic difference equations of the form given by Eq. (1). The state space for these equations is the non-negative cone $C = [0, \infty)^n$. Define the standard ordering on this cone by $x \geq y$ for $x, y \in C$ if $x_i \geq y_i$ for all i . Furthermore, $x > y$ if $x \geq y$ but $x \neq y$ and $x \gg y$ if $x_i > y_i$ for all i . Throughout, I will use $\|x\| = \max_i |x_i|$ to denote the sup norm and $\|A\| = \max_{\|x\|=1} \|Ax\|$ to denote the associated operator norm. Define the co-norm of a matrix A by $\text{co}(A) = \min_{\|x\|=1} \|Ax\|$. The co-norm is the minimal amount that the matrix A stretches a vector. Define $\log^+ x = \max\{\log x, 0\}$ to be the non-negative component of $\log x$.

For (1), there are *five standing assumptions*

A1 Uncorrelated environmental fluctuations: $\{\xi_t\}_{t=0}^\infty$ is a sequence of independent and identically distributed (i.i.d) random variables taking values in a separable metric space E (such as \mathbb{R}^k).

A2 Feedbacks depend continuously on population and environmental state: the entries of the matrix function $A_{ij} : C \times E \rightarrow [0, \infty)$ are continuous functions of population state x and the environmental state ξ .

A3 The population only experiences positive feedbacks: For all i, j and $\xi \in E$, $A_{ij}(x, \xi) \geq A_{ij}(y, \xi)$ whenever $x \geq y$.

A4 Primitivity: There exists $\tau \geq 0$ such that $A(x, \xi)^\tau \gg 0$ for all $x \gg 0$ and $\xi \in E$.

A5 Finite logarithmic moments: For all $c \geq 0$, $\mathbb{E}[\log^+ \|A(c\mathbf{1}, \xi_t)\|] < \infty$ where $\mathbf{1} = (1, 1, \dots, 1)$ is the vector of ones. There exists $c^* > 0$ such that $\mathbb{E}[\log^+ (1/\text{co}(\prod_{t=1}^\tau A(c\mathbf{1}, \xi_t)))] < \infty$ for all $c \geq c^*$.

The first assumption implies that $(X_t)_{t \geq 0}$ is a Markov chain on C and the second assumption ensures this stochastic process is Feller. The third assumption is consistent with the intent of understanding how non-negative feedbacks, in and of themselves, influence structured population dynamics. An important implication of this assumption is that the system is monotone i.e. if $X_0 > \tilde{X}_0 > 0$, then $X_t \geq \tilde{X}_t$ for all $t \geq \tau$ where X_t, \tilde{X}_t are solutions to (1) with initial conditions X_0 and \tilde{X}_0 , respectively. The fourth assumption ensures that all states in the population contribute to all other population states after τ time steps. The final assumption is met for most models and ensures that Kingman’s subadditive ergodic theorem [16] and the random Perron–Frobenius theorem [1] are applicable.

To see that these assumption include models of biological interest, here are a few examples.

Example 1 (Scalar models) Considered an unstructured population with $n = 1$ in which case $x \in [0, \infty)$. To model mate limitation, McCarthy et al. [18] considered a model where x corresponds to the density of females and, with the assumption of a 1:1 sex ratio, also equals the density of males. The probability of a female successfully mating is given by $ax/(1 + ax)$ where x is the male density and $a > 0$ determines how effectively individuals find mates. If a mated individual produces on average ξ daughters, then the population density in the next year is $\xi ax^2/(1 + ax)$.

If we allow ξ to be stochastic, then (1) is determined by $A(x, \xi) = \xi ax^2/(1 + ax)$. Allowing the ξ_t to be a log-normal would satisfy assumptions **A1–A5**.

To model predator saturation [23], let $\exp(-M/(1 + hx))$ be the probability that an individual escapes predation from a predator population with an “effective” attack rate of M and handling time h . If ξ is the number of offspring produced by an individual which escaped predation, then the population density in the next year is $\xi x \exp(-M/(1 + hx))$. Letting ξ be stochastic yields $A(x, \xi) = \xi x \exp(-M/(1 + hx))$. Allowing the ξ_t to be a log-normal would satisfy assumptions **A1–A5**.

Finally, Leibhold and Bascombe [17] used a more phenomenological model of the form $A(x, \xi) = \exp(x - C + \xi)$ where C is the critical threshold in the absence of stochasticity and ξ are normally distributed with mean zero. This model also satisfies all of the assumptions.

We can use these scalar models, which were studied by [21], to build structured models as the next two examples illustrate.

Example 2 (Spatial models) Consider a population that lives in n distinct patches. x_i is the population density in patch i . Let $C_i > 0$ be the critical threshold in patch i and ξ_i be the environmental state in patch i . Let d_{ij} be the fraction of individuals dispersing from patch j to patch i , and $D = (d_{ij})$ be the corresponding dispersal matrix. Then the spatial model is

$$A(x, \xi) = D \text{diag}(\exp(x_1 - C_1 + \xi_1), \exp(x_2 - C_2 + \xi_2), \dots, \exp(x_n - C_n + \xi_n)) \quad (2)$$

where diag denotes a diagonal matrix with the indicated diagonal elements. If D is a primitive matrix and the $\xi_t = (\xi_{1,t}, \dots, \xi_{n,t})$ are a multivariate normals with zero means, then this model satisfies the assumptions.

Example 3 (Age-structured models) Consider a population with n age classes and x_i is the density of age i individuals. Assume that final ℓ age classes reproduce i.e. ages $n - \ell + 1, n - \ell + 2, \dots, n$ reproduce. If mate limitation causes positive density dependence (see Example 1) and reproductively mature individuals mate randomly, then the fecundity of individuals in age class $n - \ell + i$ equals $f_i(x, \xi) = \xi_i a \sum_{j=n-\ell+1}^n x_j / (1 + a \sum_{j=n-\ell+1}^n x_j)$ where ξ_i is the maximal fecundity of individuals of age i and $a > 0$. Let s_i be the probability an individual survives from age $i - 1$ to age i . This yields the following nonlinear Leslie matrix model

$$A(x, \xi) = \begin{pmatrix} 0 & \dots & 0 & f_1(x, \xi) & \dots & f_\ell(x, \xi) \\ s_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & s_n & 0 \end{pmatrix}. \quad (3)$$

If $\ell \geq 2$ and $\xi_t = (\xi_{1,t}, \dots, \xi_{n,t})$ are multivariate log-normals, then this model satisfies the assumptions **A1–A5**.

3 Main Results

To state the main results, consider the linearization of (1) at the origin and near infinity. At the origin, the linearized dynamics are given by $X_{t+1} = A(0, \xi_{t+1})X_t$. Hence, the rate at which the population grows at low density is approximately given by the rate at which the random product of matrices, $A(0, \xi_t) \dots A(0, \xi_1)$, grows. Kingman’s subadditive ergodic theorem [16] implies there exists r_0 (possibly $-\infty$) such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|A(0, \xi_t) \dots A(0, \xi_1)\| = r_0 \text{ with probability one.}$$

To characterize population growth near infinity, for all $c > 0$ the subadditive ergodic theorem implies there exists an r_c such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|A(c\mathbf{1}, \xi_t) \dots A(c\mathbf{1}, \xi_1)\| = r_c \text{ with probability one.}$$

Due to our assumption that the entries of $A(x, \xi)$ are non-decreasing with respect the entries of x , r_c is non-decreasing with respect to c . Hence, the following limit exists (possibly $+\infty$)

$$r_\infty = \lim_{c \rightarrow \infty} r_c.$$

With these definitions and assumptions, the following theorem is proven in Sect. 5.

Theorem 1 *Unconditional persistence* If $r_0 > 0$, then

$$\lim_{t \rightarrow \infty} \|X_t\| = \infty \text{ with probability one whenever } X_0 \gg 0.$$

Unconditional extinction If $r_\infty < 0$, then

$$\lim_{t \rightarrow \infty} X_t = 0 \text{ with probability one.}$$

Conditional persistence and extinction If $r_0 < 0 < r_\infty$, then for all $\varepsilon > 0$ there exist $c^* > c_* > 0$ such that

$$\mathbb{P} \left[\lim_{t \rightarrow \infty} X_t = 0 \mid X_0 = x \right] \geq 1 - \varepsilon \text{ whenever } x \leq c_* \mathbf{1}$$

and

$$\mathbb{P} \left[\lim_{t \rightarrow \infty} \|X_t\| = \infty \mid X_0 = x \right] \geq 1 - \varepsilon \text{ whenever } x \geq c^* \mathbf{1}.$$

To get statements about all initial conditions with probability one in the final case, an assumption that ensures that the environmental stochasticity can drive the population to low or high densities is needed. Define $\{0, \infty\}$ to be *accessible* if for

all $c > 0$ there exists $\gamma > 0$ such that

$$\mathbb{P}\left[\left\{\text{there is } t \geq 0 \text{ such that } X_t \gg c\mathbf{1} \text{ or } X_t \ll \mathbf{1}/c\right\} \middle| X_0 = x\right] \geq \gamma$$

for all $x \gg 0$. All of the examples in Sect. 2 satisfy this accessibility condition.

Theorem 2 *If $r_0 < 0 < r_\infty$ and $\{0, \infty\}$ is accessible, then*

$$\mathbb{P}\left[\lim_{t \rightarrow \infty} \|X_t\| = \infty \text{ or } \lim_{t \rightarrow \infty} X_t = 0 \middle| X_0 = x\right] = 1.$$

Proofs of both theorems are presented in Sect. 5. The scalar version of these theorems were proven in Theorem 3.2 of [21].

4 Applications

To illustrate the applicability of the two theorems, we consider the spatial structured and age structured models introduced in Sect. 3.

Example 2 (spatially structured populations) revisited Consider the spatial structured model described in Example 2 and characterized by (2). For this model,

$$A(c\mathbf{1}, \xi) = D \text{diag}(\exp(-C_1 + \xi_1), \exp(-C_2 + \xi_2), \dots, \exp(-C_n + \xi_n)) \exp(c).$$

For simplicity, let us assume that the fraction of individuals dispersing is d and dispersing individuals land with equal likelihood on any patch (including the possibility of returning to its original patch). Then $D = (d_{ij})$ is given by $d_{ij} = d/n$ for $i \neq j$ and $d_{ii} = (1 - d) + d/n$. Assume that $d \in (0, 1]$.

I claim that $r_\infty = \infty$. Indeed, let $b = \max\{1 - d, d/n\} > 0$. Then $D \geq b\text{Id}$ where Id denotes the identity matrix and

$$\begin{aligned} \mathbb{E}[\log \|\prod_{s=1}^t A(c\mathbf{1}, \xi_s)\|] &\geq \mathbb{E}[\log \|\prod_{s=1}^t b \text{diag}(\exp(-C_1 + \xi_{1,s}), \exp(-C_2 \\ &\quad + \xi_{2,s}), \dots, \exp(-C_n + \xi_{n,s})) \exp(c)\|] \\ &\geq \mathbb{E}[\log \|\prod_{s=1}^t \text{diag}(\exp(\xi_{1,s}), \exp(\xi_{2,s}), \dots, \exp(\xi_{n,s}))\|] \\ &\quad + t(c + \log b - \max_i C_i) \\ &= \mathbb{E}[\max_i \sum_{s=1}^t \xi_{i,s}] + t(c + \log b - \max_i C_i) \\ &\geq t \left(\mathbb{E}[\xi_{1,1}] + c + \log b - \max_i C_i \right). \end{aligned}$$

Dividing by t and taking the limit as $t \rightarrow \infty$, this inequality implies that $r_c \geq \mathbb{E}[\xi_{1,1}] + c + \log b - \max_i C_i$. Hence, $r_\infty = \lim_{c \rightarrow \infty} r_c = \infty$ as claimed. Theorem 2 implies that for all $x \gg 0$, $\|X_t\| \rightarrow \infty$ with positive probability whenever $X_0 = x$.

Understanding r_0 is more challenging. However, Proposition 3 of [3] implies that r_0 varies continuously as a function of d . In the limit of $d = 0$, $D = \text{Id}$ and $r_0 = \max_i \mathbb{E}[\xi_{i,1} - C_i]$. Hence, for populations where $d \approx 0$ but $d > 0$, there are two types of dynamics. If $\mathbb{E}[\xi_{i,1}] < C_i$ for all patches (i.e. populations are unable to persist in each patch at low density), then there is a positive probability of going either asymptotically extinct or a complementary positive probability of persistence. Alternatively, if $\mathbb{E}[\xi_{i,1}] > C_i$ for at least one patch, then the population persists with probability one whenever $X_0 \gg 0$.

Now consider the case that all individuals disperse i.e. $d = 1$. Then $r_0 = \mathbb{E}[\log \frac{1}{n} \sum_i \exp(\xi_{i,1} - C_i)]$ i.e. e^{r_0} is the geometric mean of the spatial average of the $\exp(\xi_{i,1} - C_i)$. By Jensen's inequality, r_0 when $d \approx 1$ is greater than r_0 when $d \approx 0$. Hence, one can get the scenario where increasing the dispersal fraction d shifts a population from experiencing asymptotic extinction with positive probability to a population that persists with probability one. This corresponds to a positive density-dependence analog of a phenomena observed in models with negative density-dependent feedbacks [3, 14] and density-independent feedbacks [10, 15, 19, 25]. However, in these models, the long-term outcome never exhibits a mixture of extinction and persistence.

Example 3 (age-structured populations) revisited Consider the age-structured model with mate-limitation in Example 3 where there are $\ell \geq 2$ reproductive stages. If ξ_t are multivariate log-normals, then $\{0, \infty\}$ is accessible. Define

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ s_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & s_n & 0 \end{pmatrix}.$$

As $0 < s_i < 1$ for all i , the dominant eigenvalue λ of B is strictly less than one. Thus,

$$\begin{aligned} r_0 &= \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}[\log \|\prod_{s=1}^t A(0, \xi_s)\|] = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|B^t\| \\ &= \log \lambda < 0. \end{aligned}$$

As $r_0 < 0$, it follows that for all positive initial conditions there is a positive probability of asymptotic extinction (in contrast the spatial model which always has a positive probability of persistence and unbounded growth.)

To say something about persistence, assume that $\xi_{1,t}, \dots, \xi_{\ell,t}$ have the same log mean μ and non-degenerate log-covariance matrix Σ^2 . Then r_∞ is an increasing

function of μ with $\lim_{\mu \rightarrow \infty} r_\infty = \infty$ and $\lim_{\mu \rightarrow -\infty} r_\infty < 0$. Hence, there is a critical μ , call it μ^* , such that the population goes asymptotically extinct with probability one whenever $\mu < \mu^*$ and the population persists with positive probability whenever $\mu > \mu^*$.

5 Proofs

First, I prove Theorem 1. Assume $r_0 > 0$ and $X_0 = x_0 \gg 0$. As the entries of $A(x, \xi)$ are non-decreasing functions of x ,

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \log \|X_t\| &= \liminf_{t \rightarrow \infty} \frac{1}{t} \log \left\| \prod_{s=1}^t A(X_{s-1}, \xi_s) x_0 \right\| \\ &\geq \liminf_{t \rightarrow \infty} \frac{1}{t} \log \left\| \prod_{s=1}^t A(0, \xi_s) x_0 \right\| \\ &= r_0 > 0 \text{ with probability one.} \end{aligned}$$

In particular, $\lim_{t \rightarrow \infty} \|X_t\| = \infty$ with probability one as claimed.

Next, assume that $r_\infty < 0$. Given any $X_0 = x_0 \gg 0$, choose $c > 0$ such that $c\mathbf{1} \geq x_0$ and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \left\| \prod_{s=1}^t A(c\mathbf{1}, \xi_s) \right\| \leq r_\infty/2 \text{ with probability one.}$$

Then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|X_t\| &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left\| \prod_{s=1}^t A(c\mathbf{1}, \xi_s) x_0 \right\| \\ &\leq r_\infty/2 < 0 \text{ with probability one.} \end{aligned}$$

In particular, $\lim_{t \rightarrow \infty} X_t = 0$ with probability one as claimed.

Finally, assume that $r_\infty > 0$ and $r_0 < 0$. As the entries of A are non-decreasing in x , there exists $c > 0$ such that $A(c\mathbf{1}, \xi) \leq A(0, \xi) \exp(-r_0/2)$ for $\xi \in E$. Hence,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \left\| \prod_{s=1}^t A(c\mathbf{1}, \xi_s) \right\| \leq r_0/2 < 0 \text{ with probability one.} \quad (4)$$

Define the random variable

$$R = \sup_{t \geq 1} \left\| \prod_{s=1}^t A(c\mathbf{1}, \xi_s) \right\|.$$

Equation (4) implies that $R < \infty$ with probability one. For all $k > 0$, define the event $\mathcal{E}_k = \{R \leq k\}$. For $x_0 \leq c\mathbf{1}/k$ and $X_0 = x_0$, I claim that $X_t \leq c\mathbf{1}$ for all $t \geq 0$ on the event \mathcal{E}_k . I prove this claim by induction. $X_0 \leq c\mathbf{1}$ by assumption. Suppose that $X_s \leq c\mathbf{1}$ for $0 \leq s \leq t - 1$. Then

$$\begin{aligned} \|X_t\| &= \left\| \prod_{s=1}^t A(X_{s-1}, \xi_s) x_0 \right\| \\ &\leq \left\| \prod_{s=1}^t A(c\mathbf{1}, \xi_s) c\mathbf{1}/k \right\| \text{ by induction and monotonicity} \\ &\leq \left\| \prod_{s=1}^t A(c\mathbf{1}, \xi_s) \right\| c/k \leq Rc/k \text{ by the definition of } R \text{ and } x \\ &\leq c \text{ on the event } \mathcal{E}_k. \end{aligned}$$

This completes the proof of the claim that $X_t \leq c\mathbf{1}$ for all $t \geq 0$ on the event \mathcal{E}_k . It follows that on the event \mathcal{E}_k and $X_0 = x \leq c\mathbf{1}/k$ that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|X_t\| &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left\| \prod_{s=1}^t A(c\mathbf{1}, \xi_s) \right\| c \\ &\leq r_0/2 < 0 \text{ almost surely.} \end{aligned}$$

In particular, $\lim_{t \rightarrow \infty} X_t = 0$ almost sure on the event \mathcal{E}_k . As the events \mathcal{E}_k are increasing with k , $\lim_{k \rightarrow \infty} \mathbb{P}[\mathcal{E}_k] = \mathbb{P}[\cup_k \mathcal{E}_k] = \mathbb{P}[R < \infty] = 1$. Therefore, given $\varepsilon > 0$, there exists k such that $\mathbb{P}[\mathcal{E}_k] > 1 - \varepsilon$. For this k , $x_0 \leq c\mathbf{1}/k$ and $X_0 = x_0$,

$$\mathbb{P}[\lim_{t \rightarrow \infty} X_t = 0 | X_0 = x_0] \geq \mathbb{P}[\mathcal{E}_k] \geq 1 - \varepsilon.$$

To show convergence to ∞ with positive probability when $r_\infty > 0$, choose $c \geq c^*$ sufficiently large so that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \left\| \prod_{s=1}^t A(c\mathbf{1}, \xi_s) \right\| \geq r_\infty/2 > 0 \text{ with probability one.}$$

By the Random Perron–Frobenius theorem [1, Theorem 3.1 and Remark (ii) on pg. 878],

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \left(e_i^T \prod_{s=1}^t A(c\mathbf{1}, \xi_s) e_j \right) \geq r_\infty/2 > 0 \text{ with probability one.} \quad (5)$$

for all elements e_i, e_j of the standard basis of \mathbb{R}^n and where T denotes the transpose of a vector. Equation (5) implies that all of the entries of $\prod_{s=1}^t A(c\mathbf{1}, \xi_s)$ grow exponentially in time at rate greater than $r_\infty/2$ with probability one.

Define

$$R_\infty = \inf_{t \geq 1, 1 \leq i \leq n} e_i^T \prod_{s=1}^t A(c\mathbf{1}, \xi_s) c\mathbf{1}.$$

By (5) and the primitivity assumption **A4**, $R_\infty > 0$ with probability one. Define the events

$$\mathcal{F}_k = \{R_\infty > 1/k\} \text{ for } k \geq 1.$$

Now, suppose that $X_0 = x_0 \geq c\mathbf{1}k$. I claim that $X_t \geq c\mathbf{1}$ for all $t \geq 0$ on the event \mathcal{F}_k . $X_0 \geq c\mathbf{1}$ by the choice of x_0 . Assume that $X_s \geq c\mathbf{1}$ for $0 \leq s \leq t-1$. Then

$$\begin{aligned} X_t &= \prod_{s=1}^t A(X_{s-1}, \xi_s) x_0 \\ &\geq \prod_{s=1}^t A(c\mathbf{1}, \xi_s) x_0 \text{ by inductive hypothesis} \\ &\geq R_\infty c\mathbf{1}k \text{ by definition of } R_\infty \text{ and } x_0 \\ &\geq c\mathbf{1} \text{ on the event } \mathcal{F}_k. \end{aligned}$$

Equation (5) implies that on the event \mathcal{F}_k

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \|X_t\| \geq r_0/2 \text{ almost surely.}$$

Hence, $\lim_{t \rightarrow \infty} \|X_t\| = \infty$ almost surely on the event \mathcal{F}_k . As \mathcal{F}_k are an increasing set of events, $\mathbb{P}[R_\infty > 0] = \mathbb{P}[\cup_{t \geq 1} \mathcal{F}_k] = 1$. For any $\varepsilon > 0$ there is $k \geq 1$ such that $\mathbb{P}[\mathcal{F}_k] \geq 1 - \varepsilon$. Hence, for this k and $X_0 = x \geq ck\mathbf{1}$,

$$\mathbb{P}[\lim_{t \rightarrow \infty} \|X_t\| = \infty | X_0 = x] \geq 1 - \varepsilon.$$

This completes the proof of Theorem 1.

The proof of Theorem 2 follows from Theorem 1 and the following proposition.

Proposition 1 *Assume $\{0, \infty\}$ is accessible. Let $c > 0$ and $\delta \in [0, 1)$ be such that*

$$\mathbb{P} \left[\lim_{t \rightarrow \infty} X_t = 0 | X_0 = x \right] \geq 1 - \delta \text{ whenever } x \leq \mathbf{1}/c$$

and

$$\mathbb{P} \left[\lim_{t \rightarrow \infty} X_t = \infty | X_0 = x \right] \geq 1 - \delta \text{ whenever } x \geq c\mathbf{1}.$$

Then

$$\mathbb{P} \left[\lim_{t \rightarrow \infty} X_t = \infty \text{ or } \lim_{t \rightarrow \infty} X_t = 0 | X_0 = x \right] = 1 \text{ whenever } x \gg 0.$$

Proof Define the event

$$C = \left\{ \lim_{t \rightarrow \infty} X_t = \infty \text{ or } \lim_{t \rightarrow \infty} X_t = 0 \right\}.$$

For any $x \in C$, define $\mathbb{P}_x[\mathcal{E}] = \mathbb{P}[\mathcal{E} | X_0 = x]$ (respectively, $\mathbb{E}_x[Z] = \mathbb{E}[Z | X_0 = x]$) for any event \mathcal{E} (respectively, random variable Z) in the σ -algebra generated by $\{X_0 = x, X_1, X_2, \dots\}$. Furthermore, define $I_{\mathcal{E}}$ to be random variable that equals 1 on the event \mathcal{E} and 0 otherwise.

Define the stopping time

$$S = \inf\{t \geq 0 : X_t \geq c\mathbf{1} \text{ or } X_t \leq \mathbf{1}/c\}.$$

Since $\{0, \infty\}$ is accessible, there exists $\gamma > 0$ such that $\mathbb{P}_x[S < \infty] > \gamma$ for all $x \gg 0$. Let $I_{\{S < \infty\}}$ equal 1 if $S < \infty$ and 0 otherwise. The strong Markov property implies that for all $x \gg 0$

$$\begin{aligned} \mathbb{P}_x[C] &= \mathbb{E}_x \left[\mathbb{P}_{X_S}[C] I_{\{S < \infty\}} \right] + \mathbb{E}_x \left[\mathbb{P}_{X_S}[C] I_{\{S = \infty\}} \right] \\ &= \mathbb{E}_x \left[\mathbb{P}_{X_S}[C] I_{\{S < \infty\}} \right] \\ &\geq (1 - \delta)\gamma. \end{aligned}$$

Let \mathcal{F}_t be the σ -algebra generated by $\{X_1, \dots, X_t\}$. The Lévy zero-one law implies that for all $x \gg 0$, $\lim_{t \rightarrow \infty} \mathbb{E}_x [I_C | \mathcal{F}_t] = I_C$ almost surely. On the other hand, the Markov property implies that $\mathbb{E}_x [I_C | \mathcal{F}_t] = \mathbb{P}_{X_t}[C] \geq (1 - \delta)\gamma$ for all $x \gg 0$. Hence $\mathbb{P}_x[C] = 1$ for all $x \gg 0$. □

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